Estimation of Volatility Functionals in the Simultaneous Presence of Microstructure Noise and Jumps*

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Abstract

We propose a new concept of modulated bipower variation for diffusion models with microstructure noise. We show that this method provides simple estimates for such important quantities as integrated volatility or integrated quarticity. Under mild conditions the consistency of modulated bipower variation is proven. Under further assumptions we prove stable convergence of our estimates with the optimal rate $n^{-\frac{1}{4}}$. Moreover, we construct estimates which are robust to finite activity jumps.

Keywords: Bipower Variation; Central Limit Theorem; Finite Activity Jumps; High-Frequency Data; Integrated Volatility; Microstructure Noise; Semimartingale Theory; Subsampling.

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1 Introduction

Continuous time stochastic models represent a widely accepted class of processes in mathematical finance. Ito diffusions, which are characterised by the equation

$$X_t = X_0 + \int_0^t a_s ds + \int_0^t \sigma_s dW_s \ , \tag{1.1}$$

are commonly used for modeling the dynamics of interest rates or stock prices. Here W denotes a Brownian motion, a is a locally bounded predictable drift function and σ is a càdlàg volatility function. A key issue in econometrics is the estimation (and forecasting) of quadratic variation of X

$$IV = \int_0^1 \sigma_s^2 ds \ ,$$

which is known as integrated volatility or integrated variance in the econometric literature. In the last years the availability of high frequency data on financial markets has motivated a huge number of publications devoted to measurement of integrated volatility. The most conspicuous idea of estimation of integrated volatility is the realised volatility (RV), which has been proposed by Andersen, Bollerslev, Diebold & Labys (2001) and Barndorff-Nielsen & Shephard (2002). RV is the sum of squared increments over non-overlapping intervals within a sampling period. The consistency result justifying this estimator is a simple consequence of the definition of quadratic variation. Theoretical and empirical properties of the realised volatility have been studied in numerous articles (see Jacod (1994), Jacod & Protter (1998), Andersen, Bollerslev, Diebold & Labys (2001), Barndorff-Nielsen & Shephard (2002) among many others).

More recently, the concept of realised bipower variation has built a non-parametric framework for backing out several variational measures of volatility (see, e.g., Barndorff-Nielsen & Shephard (2004) or Barndorff-Nielsen, Graversen, Jacod, Podolskij & Shephard (2006)), which has lead to a new development in econometrics. Realised bipower variation, which is defined by

$$BV(X,r,l)_n = n^{\frac{r+l}{2}-1} \sum_{i=1}^{n-1} |\Delta_i^n X|^r |\Delta_{i+1}^n X|^l \ , \tag{1.2}$$

with $\Delta_i^n X = X_{i/n} - X_{(i-1)/n}$ and $r, l \geq 0$, provides a whole class of estimators for different (integrated) powers of volatility. Another important feature of realised bipower variation is its robustness to finite activity jumps when estimating integrated volatility. This property has been used to construct tests for jumps (see Barndorff-Nielsen & Shephard (2005) or Christensen & Podolskij (2006b)).

However, in finance it is widely accepted that the true price process is contaminated by microstructure effects, such as price discreteness or bid-ask spreads, among others. This invalidates the asymptotic properties of RV, and in the presence of microstructure noise RV is both biased and inconsistent (see Bandi & Russel (2004) or Hansen & Lunde (2006) among others). Nowadays there exist two concurrent methods of estimating integrated volatility in the presence of i.i.d. noise. Zhang, Mykland & Ait-Sahalia (2005) have proposed to use a *two scale* estimator, which is based on a subsampling procedure (a *multiscale* estimator proposed by Zhang (2006) is more efficient than a *two scale* estimator). Another method is a *realised kernel* estimator which has been proposed by Barndorff-Nielsen, Hansen, Lunde & Shephard (2006). Both methods provide consistent estimates of integrated volatility in the presence of i.i.d. noise and achieve the optimal rate $n^{-\frac{1}{4}}$. However, these procedures can not be generalised in order to obtain estimators of other (integrated) powers of volatility, such as integrated quarticity, which is defined by

$$IQ = \int_0^1 \sigma_s^4 ds.$$

This quantity is of particular interest, because, properly scaled, it occurs as conditional variance in the central limit theorem for RV and has to be estimated. Moreover, both methods are not robust to jumps in the price process.

In this paper we propose a new concept of modulated bipower variation (MBV) for diffusion models with (i.i.d.) microstructure noise. The novelty of this concept is twofold. First, this method provides a class of estimates for arbitrary integrated powers of volatility. Second, modulated multipower variation, which is a direct generalisation of MBV, turns out to be robust to finite activity jumps. In particular, we construct estimators of IV and IQ which are robust to finite activity jumps. To the best of our knowledge these are the first consistent estimates of IV and IQ when both microstructure noise and jumps are present. An easy implementation of MBV is another nice feature of our method.

This paper is organised as follows. In Section 2 we state the basic notations and definitions. In Section 3 we show the consistency of our estimators and prove a central limit theorem for its normalized versions with an optimal rate $n^{-\frac{1}{4}}$. In particular, we construct some new estimators of integrated volatility and integrated quarticity, and present the corresponding asymptotic theory. Moreover, we demonstrate how the assumptions on the noise process can be relaxed. Section 4 illustrates the finite sample properties of our approach by means of a Monte Carlo study. Some conclusions and directions for future research are highlighted in Section 5. Finally, we present the proofs in the Appendix.

2 Basic notations and definitions

We consider the process Y, observed at time points $t_i = i/n$, $i = 0, \dots, n$. Y is defined on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,1]}, P)$ and exhibits a decomposition

$$Y = X + U\ , \tag{2.1}$$

where X is a diffusion process defined by (1.1), and U is an i.i.d. symmetric noise process with

$$EU = 0\ , \qquad EU^2 = \omega^2. \tag{2.2}$$

Further, we assume that X and U are independent.

The core of our approach is the following class of statistics

$$MBV(Y,r,l)_n = n^{\frac{(r+l)}{4}-\frac{1}{2}} \sum_{m=1}^{M} |\bar{Y}_m^{(K)}|^r |\bar{Y}_{m+1}^{(K)}|^l \qquad r,l \geq 0 \ , \tag{2.3}$$

$$\bar{Y}_m^{(K)} = \frac{1}{\frac{n}{M} - K + 1} \sum_{i=\frac{(m-1)n}{M}}^{\frac{mn}{M}-K} \Big(Y_{\frac{i+K}{n}} - Y_{\frac{i}{n}} \Big). \tag{2.4}$$

Here $M = M(n)$ and $K = K(n)$ are two natural numbers approaching infinity as $n \to \infty$ which will be chosen below. Clearly, the constants M, K must satisfy

$$K \leq \frac{n}{M} \ , \tag{2.5}$$

because otherwise the defintion in (2.4) makes no sense. Note that $\bar{Y}_m^{(K)}$ is the mean of all increments of length $\frac{K}{n}$ within the interval $[\frac{m-1}{M}, \frac{m}{M}]$.

Remark 1 *In the definition (2.3) $m+1$ can be replaced by $m+q$ for any fixed natural q. Such procedure has been suggested for $BV(X,r,l)$ by Andersen, Bollerslev & Diebold (2006) and Barndorff-Nielsen & Shephard (2006). Huang & Tauchen (2005) show by empirical studies that extra lagging reduces the impact of microstructure noise on $BV(X,r,l)$. This may also cause an improvement of empirical behaviour of $MBV(Y,r,l)_n$, but the asymptotic results are not affected by this change.*

The intuition behind the estimator defined by (2.3) can be explained as follows. The constants K and M control the stochastic order of the term $\bar{Y}_m^{(K)}$. In particular, when $\frac{n}{M} - K$ converges to infinity we have

$$\bar{U}_m^{(K)} = O_p\left(\sqrt{\frac{1}{\frac{n}{M} - K}}\right) \ , \tag{2.6}$$

$$\bar{X}_m^{(K)} = O_p\Big(\sqrt{\frac{K}{n}}\Big). \tag{2.7}$$

If

$$K = c_1 n^{\frac{1}{2}} \ , \qquad M = \frac{n}{c_2 K} \tag{2.8}$$

for some constants $c_1 > 0$ and $c_2 > 1$ (which will be chosen later), the stochastic orders of the quantities in (2.6) and (2.7) are balanced, and we obtain

$$\bar{Y}_m^{(K)} = O_p(n^{-\frac{1}{4}}) \ , \tag{2.9}$$

which explains the normalizing factor in (2.3).

More generally, we define the modulated multipower variation by setting

$$MMV(Y,r_1,\dots,r_k)_n = n^{\frac{r_+}{4}-\frac{1}{2}} \sum_{m=1}^{M-k+1} \prod_{j=1}^{k} |\bar{Y}^{(K)}_{m+k-1}|^{r_j} \ ,$$

where k is a fixed natural number, $r_j \geq 0$ for all j and $r_+ = r_1 + \cdots + r_k$. This type of construction has been intensively used in a pure Ito diffusion framework (see, for instance, Barndorff-Nielsen & Shephard (2007) or Christensen & Podolskij (2006b) among others). Later on we will show that the modulated multipower variation, for an appropriate choice of k and $r_1,\dots,r_k$, turns out to be robust to finite activity jumps when estimating arbitrary powers of volatility.

In the sequel we mainly focus on the asymptotic theory of the modulated bipower variation, but we also state the corresponding results for $MMV(Y,r_1,\dots,r_k)_n$ for the sake of completeness.

3 Asymptotic theory

In this section we study the asymptotic behaviour of the class of estimators $MBV(Y,r,l)_n$, $r,l \geq 0$. Before we state the main results of this section we introduce the following notation:

$$\mu_r = E[|z|^r], \qquad z \sim N(0,1). \tag{3.1}$$

3.1 Consistency

Theorem 1 *Assume that $E|U|^{2(r+l)+\epsilon} < \infty$ for some $\epsilon > 0$. If M and K satisfy (2.8) then the convergence in probability*

$$MBV(Y,r,l)_n \xrightarrow{P} MBV(Y,r,l) = \frac{\mu_r \mu_l}{c_1 c_2} \int_0^1 (\nu_1 \sigma_u^2 + \nu_2 \omega^2)^{\frac{r+l}{2}} \ du \tag{3.2}$$

holds. The constants ν_1 and ν_2 are given by

$$\nu_1 = \frac{c_1(3c_2 - 4 + (2-c_2)^3 \vee 0)}{3(c_2-1)^2} \ ,$$

$$\nu_2 = \frac{2((c_2-1)\wedge 1)}{c_1(c_2-1)^2}. \tag{3.3}$$

It is remarkable that the limit $MBV(Y,r,l)$ in (3.2) depends only on the second moment ω^2 of U, and no higher moments are involved. This can be illustrated as follows. Observe that due to the choice of the constants in (2.8) we have

$$n^{\frac{1}{4}} \bar{U}^{(K)}_m \xrightarrow{\mathcal{D}} \mathcal{N}(0, \nu_2 \omega^2) \ , \tag{3.4}$$

which is justified by a standard central limit theorem. Under the regularity condition of Theorem 1 the moments of $\bar{U}^{(K)}_m$ can be (asymptotically) replaced by the corresponding moments of the normal distribution in (3.4), which only depend on ω^2.

In fact, the estimation of higher moments of U turns out to be difficult in practice, because they are extremely small. Note, for instance, that the asymptotic results for the twoscale (multiscale) estimator of integrated volatility depend on the fourth moment of U. Since only the second moment ω^2 is involved in our approach, we do not face these problems.

Finally, we present the convergence in probability of the modulated multipower variation $MMV(Y, r_1, \ldots, r_k)_n$.

Theorem 2 *Assume that $E|U|^{2r_+ + \epsilon} < \infty$ for some $\epsilon > 0$. If M and K satisfy (2.8) then the convergence in probability*

$$MMV(Y, r_1, \ldots, r_k)_n \xrightarrow{P} MMV(Y, r_1, \ldots, r_k) = \frac{\mu_{r_1} \cdots \mu_{r_k}}{c_1 c_2} \int_0^1 (\nu_1 \sigma_u^2 + \nu_2 \omega^2)^{\frac{r_+}{2}} \, du \tag{3.5}$$

holds.

3.1.1 Consistent estimates of integrated volatility and integrated quarticity

Theorem 1 shows that $MBV(Y, r, l)_n$ is inconsistent when estimating arbitrary (integrated) powers of volatility. Though, when $r + l$ is an even number (this condition is satisfied for the most interesting cases) a slight modification of $MBV(Y, r, l)_n$ turns out to be consistent. Let us illustrate this procedure by providing consistent estimates for integrated volatility and integrated quarticity.

As already mentioned in Zhang, Mykland & Ait-Sahalia (2005) the statistic

$$\hat{\omega}^2 = \frac{1}{2n} \sum_{i=1}^{n} |Y_{\frac{i}{n}} - Y_{\frac{i-1}{n}}|^2 \tag{3.6}$$

is a consistent estimator of the quantity ω^2 with the convergence rate $n^{-\frac{1}{2}}$. Consequently, we obtain the convergence in probability

$$MRV(Y)_n := \frac{c_1 c_2 MBV(Y, 2, 0)_n - \nu_2 \hat{\omega}^2}{\nu_1} \xrightarrow{P} \int_0^1 \sigma_u^2 \, du \tag{3.7}$$

as a direct application of Theorem 1 and (3.6).

Now we are in a position to construct a consistent estimator of integrated quarticity. It follows from (3.7) and Theorem 1 that

$$MRQ(Y)_n := \frac{\frac{c_1 c_2}{3} MBV(4, 0)_n - 2\nu_1 \nu_2 \hat{\omega}^2 MRV(Y)_n - \nu_2^2 (\hat{\omega}^2)^2}{\nu_1^2} \xrightarrow{P} \int_0^1 \sigma_u^4 \, du. \tag{3.8}$$

Note, however, that Theorem 1 gives a whole class of new estimators of integrated volatility and integrated quarticity.

Remark 2 *The constant ν_1 corresponds to the second moment of the term $n^{\frac{1}{4}} \bar{W}_m^{(K)}$, where W is a Brownian motion. More precisely, we have*

$$n^{\frac{1}{4}} \bar{W}_m^{(K)} \sim N(0, \nu_1^{(n)}) ,$$

with

$$\nu_1^{(n)} = \nu_1 + \frac{(3-c_2) \wedge \frac{1}{c_2-1}}{(c_2-1)\sqrt{n}} + O(\frac{1}{n}). \tag{3.9}$$

Clearly, it holds that $\nu_1^{(n)} \to \nu_1$. *However, we can reduce the bias of the estimates* $MRV(Y)_n$ *and* $MRQ(Y)_n$ *by replacing* ν_1 *by* $\nu_1^{(n)}$.

3.1.2 Robustness to finite activity jumps

As already mentioned in the introduction one of our main goals is finding consistent estimates of volatility functionals when both microstructure noise and jumps are present. For this purpose we consider the model

$$Z = Y + J \ , \tag{3.10}$$

where Y is a noisy diffusion process defined by (2.1) and J denotes a finite activity jump process, i.e. J exhibits finitely many jumps on compact intervals. Typical examples of a finite activity jump process are compound Poisson processes.

The next result gives us conditions on $r_1, \ldots, r_k$ under which the modulated multipower variation $MMV(Z, r_1, \ldots, r_k)_n$ is robust to finite activity jumps.

Proposition 3 *If the assumptions of Theorem 2 are satisfied,* $\max(r_1, \ldots, r_k) < 2$ *and* Z *is of the form (3.10) then we have*

$$MMV(Z, r_1, \ldots, r_k)_n \xrightarrow{P} MMV(Y, r_1, \ldots, r_k) \ , \tag{3.11}$$

where $MMV(Y, r_1, \ldots, r_k)$ *is given by (3.5).*

Proposition 3 is shown by the same methods as the corresponding result in the noiseless model (i.e. $U = 0$). We refer to Barndorff-Nielsen, Shephard & Winkel (2006) for more details.

Now we can construct consistent estimates for integrated volatility and integrated quarticity which are robust to noise and finite activity jumps. As a direct consequence of Proposition 3 the convergence in probability

$$MBV(Z)_n := \frac{\frac{c_1 c_2}{\mu_1^2} MBV(Z,1,1)_n - \nu_2 \hat{\omega}^2}{\nu_1} \xrightarrow{P} \int_0^1 \sigma_u^2 \, du \tag{3.12}$$

holds. Similar to the previous subsection, a robust (tripower) estimate of the integrated quarticity is given by

$$MTQ(Z)_n := \frac{\frac{c_1 c_2}{\mu_{2/3}^3} MMV(Z, \frac{4}{3}, \frac{4}{3}, \frac{4}{3})_n - 2\nu_1 \nu_2 \hat{\omega}^2 MBV(Z)_n - \nu_2^2 (\hat{\omega}^2)^2}{\nu_1^2} \xrightarrow{P} \int_0^1 \sigma_u^4 \, du. \tag{3.13}$$

Remark 3 *Recall that the realised volatility* RV *converges in probability to integrated volatility plus the sum of squared jumps in the jump-diffusion model. It is interesting to see that the* *presence of jumps destroys the consistency of the estimator $MRV(Z)_n$, which can be interpreted as an analogue of RV. To show this let us consider a simple model*

$$Z = J$$

(i.e. $X = U = 0$), where J is a (deterministic) jump process that possesses one jump of size 1 at point $t_0 = \frac{1}{2}$. Moreover, we set $c_2 = 2$ and $c_1 = 1$ for simplicity. For a subsequence $M_k = 2^k$ (that is $n_k = 2^{2(k+1)}$) the point t_0 is located on the boundary of some interval $[\frac{m-1}{M}, \frac{m}{M}]$, and we have

$$MBV(Z, 2, 0)_{n_k} \xrightarrow{P} 0.$$

When we use a subsequence $M_k = 3^k$ (that is $n_k = 4 \cdot 3^{2k}$) the point t_0 lies in the middle of some interval $[\frac{m-1}{M}, \frac{m}{M}]$, and we obtain the convergence

$$MBV(Z, 2, 0)_{n_k} \xrightarrow{P} 1.$$

Consequently, the statistic $MRV(Z)_n$ does not converge in probability when there are jumps.

In contrast to our approach the multiscale estimator of Zhang (2006) and the realised kernel estimator of Barndorff-Nielsen, Hansen, Lunde & Shephard (2006) converge in probability to the quadratic variation of the jump-diffusion process $X + J$ (in the presence of noise). In principle, it is possible to test for jumps in the noisy model by comparing the multiscale estimator or the realised kernel estimator with the robust statistic $MBV(Z, 1, 1)_n$ (see, for instance, Barndorff-Nielsen & Shephard (2006) or Christensen & Podolskij (2006) for more details on such tests in the noiseless models), although we will not further discuss this idea in the paper.

Another important object of study is the impact of infinite activity jumps on the modulated bipower (multipower) variation. Such studies can be found in Barndorff-Nielsen, Shephard & Winkel (2006), Woerner (2006) and in a recent paper of Ait-Sahalia & Jacod (2006) for the noiseless framework. We are convinced that similar results hold also for the noisy model, although a more detailed analysis is required.

3.1.3 Relaxing the assumptions on the noise process U

So far we assumed that U is an i.i.d. sequence and is independent of the diffusion X. Hansen & Lunde (2006) have reported that both assumptions are somewhat unrealistic for ultra-high frequency data. In the following we demonstrate how these conditions can be relaxed.

First, note that the i.i.d assumption is not essential to guarantee the stochastic order of $\bar{U}_m^{(K)}$ in (2.6). When we assume, for instance, that U is a q-dependent sequence, the result of Theorem 1 holds, although higher order autocorrelations of U appear in the limit. In this case we require a stationarity condition on U for the estimation of the autocorrelations and a bias-correction of the limit in (3.2).

Further, by using other constants M and K the influence of the noise process U can be made negligible, and independence between X and U is not required. (2.6) and (2.7) imply that in particular, when we set

$$K = c_1 n^{\frac{1}{2}+\gamma}\ , \qquad M = \frac{n}{c_2 K} \tag{3.14}$$

for some $0 < \gamma < \frac{1}{2}$, the diffusion process X dominates the noise process U. More precisely, the convergence in probability

$$n^{\frac{(1-2\gamma)(r+l)}{4} - \frac{1-2\gamma}{2}} \sum_{m=1}^{M} |\bar{Y}_m^{(K)}|^r |\bar{Y}_{m+1}^{(K)}|^l \xrightarrow{P} \frac{\mu_r \mu_l \nu_1^{\frac{r+l}{2}}}{c_1 c_2} \int_0^1 |\sigma_u|^{r+l}\, du \tag{3.15}$$

holds. The convergence in (3.15) has another useful side effect. It provides consistent estimates for arbitrary integrated powers of volatility. However, since the diffusion process X dominates the noise process U, the above choice of K and M leads to a slower rate of convergence.

3.2 Central limit theorems

In this subsection we present the central limit theorems for a normalized version of $MBV(Y, r, l)_n$. For this purpose we need a structural assumption on the process σ.

(V): The volatility function σ satisfies the equation

$$\sigma_t = \sigma_0 + \int_0^t a'_s ds + \int_0^t \sigma'_{s-} dW_s + \int_0^t v'_{s-} dV_s. \tag{3.16}$$

Here a',σ' and v' are adapted càdlàg processes, with a' also being predictable and locally bounded, and V is a new Brownian motion independent of W.

Condition (V) is a standard assumption that is required for the proof of the central limit theorem for the pure diffusion part X (see e.g., Barndorff-Nielsen, Graversen, Jacod, Podolskij & Shephard (2006) or Christensen & Podolskij (2006a,b)). When X is a unique strong solution of a stochastic differential equation then under some smoothness assumption on the volatility $\sigma_t = \sigma(t, X_t)$ condition (V) (with $v'_s = 0$ for all s) is a simple consequence of Ito's formula. Therefore, assumption (V) is fulfilled for many widely used financial models (see Black & Scholes (1973), Vasicek (1977), Cox, Ingersoll & Ross (1980) or Chan, Karolyi, Longstaff & Sanders (1992) among others).

For technical reasons we require a further structural assumption on the noise process U. We assume that the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,1]}, P)$ supports another Brownian motion $B = (B_t)_{t\in[0,1]}$ that is independent of the diffusion process X, such that the representation

$$U_i = \sqrt{n}\omega(B_{\frac{i}{n}} - B_{\frac{i-1}{n}}) \tag{3.17}$$

holds.

Remark 4 *Condition (3.17) ensures that both processes X and U are measurable with respect to the same type of filtration. This assumption enables us to use the central limit theorems for high frequency observations (see Jacod & Shiryaev (2003)). The same assumption has already been used in Gloter & Jacod (2001a) and Gloter & Jacod (2001b).*

The normal distribution of the noise induced by (3.17) is not crucial for our asymptotic theory, and other functions of rescaled increments of B can be considered. Of course, this leads to a slight modification of the central limit theorems presented below.

In the central limit theorems which will be demonstrated below we use the concept of stable convergence of random variables. Let us shortly recall the definition. A sequence of random variables G_n converges stably in law with limit G (throughout this paper we write $G_n \xrightarrow{\mathcal{D}_{st}} G$), defined on an appropriate extension $(\Omega', \mathcal{F}', P')$ of a probability space $(\Omega, \mathcal{F}, P)$, if and only if for any $\mathcal{F}$-measurable and bounded random variable H and any bounded and continuous function g the convergence

$$\lim_{n\to\infty} E[Hg(G_n)] \;=\; E[Hg(G)]$$

holds. This is obviously a slightly stronger mode of convergence than convergence in law (see Renyi (1963) or Aldous & Eagleson (1978) for more details on stable convergence).

Now we present a central limit theorem for the statistic $MBV(Y, r, l)_n$.

Theorem 4 *Assume that U is of the form (3.17) and condition (V) is satisfied. If M and K satisfy (2.8), and*

1. *$r, l \in (1, \infty) \cup \{0\}$ or*

2. *r or $l \in (0, 1]$, and $\sigma_s \neq 0$ for all s,*

then we have

$$n^{\frac{1}{4}}\Big(MBV(Y, r, l)_n - MBV(Y, r, l)\Big) \xrightarrow{\mathcal{D}_{st}} L(r, l) \ ,$$

where $L(r, l)$ is given by

$$L(r, l) = \sqrt{\frac{\mu_{2r}\mu_{2l} + 2\mu_r\mu_l\mu_{r+l} - 3\mu_r^2\mu_l^2}{c_1 c_2}} \int_0^1 (\nu_1\sigma_u^2 + \nu_2\omega^2)^{\frac{r+l}{2}} \; dW_u'. \tag{3.18}$$

Here W' denotes another Brownian motion defined on an extension of the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,1]}, P)$ and is independent of the σ-field $\mathcal{F}$.

Note that $L(r, l)$ defined by (3.18) depends only on the second moment ω^2 of U. This is only partially a consequence of the representation (3.17)! In fact, when r and l are even the conditional variance of $L(r, l)$ is not affected by the distribution of U. This can be explained by the weak convergence in (3.4) using the same arguments as presented in Section 3.1.

Since $\hat{\omega}^2 - \omega^2 = O_p(n^{-\frac{1}{2}})$ we obtain the central limit theorems for the estimates $MRV(Y)_n$ and $MBV(Y)_n$ defined by (3.7) and (3.12), respectively, as a direct consequence of Theorem 4.

Corollary 1 *Assume that U is of the form (3.17) and condition (V) is satisfied. If M and K satisfy (2.8) then we have*

$$n^{\frac{1}{4}}\Big(MRV(Y)_n - \int_0^1 \sigma_u^2\, du\Big) \xrightarrow{\mathcal{D}_{st}} \frac{\sqrt{2c_1c_2}}{\nu_1}\int_0^1 (\nu_1\sigma_u^2 + \nu_2\omega^2)\, dW'_u\ , \tag{3.19}$$

where W' is another Brownian motion defined on an extension of the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,1]}, P)$ and is independent of the σ-field $\mathcal{F}$.

Corollary 2 *Assume that U is of the form (3.17) and condition (V) is satisfied. If M and K satisfy (2.8), and $\sigma_s \neq 0$ for all s, then we have*

$$n^{\frac{1}{4}}\Big(MBV(Y)_n - \int_0^1 \sigma_u^2\, du\Big) \xrightarrow{\mathcal{D}_{st}} \sqrt{\frac{c_1c_2(\mu_2^2 + 2\mu_1^2\mu_2 - 3\mu_1^4)}{\mu_1^4\nu_1^2}}\int_0^1 (\nu_1\sigma_u^2 + \nu_2\omega^2)\, dW'_u\ , \tag{3.20}$$

where W' is another Brownian motion defined on an extension of the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,1]}, P)$ and is independent of the σ-field $\mathcal{F}$.

Now let us demonstrate how Corollary 1 and 2 can be applied in order to obtain confidence intervals for the integrated volatility. Note that the central limit theorem in (3.19) is not feasible yet. Nevertheless, we can easily obtain a feasible version of Corollary 1. Since the Brownian motion W' is independent of the volatility process σ, the limit defined by (3.19) has a mixed normal distribution with conditional variance

$$\beta^2 = \frac{2c_1c_2}{\nu_1^2}\int_0^1 (\nu_1\sigma_u^2 + \nu_2\omega^2)^2\, du.$$

By an application of Theorem 1 and (3.6) the statistic

$$\beta_n^2 = 2c_1c_2 MRQ(Y)_n + \frac{4c_1c_2\nu_2}{\nu_1}\hat{\omega}^2 MRV(Y)_n + \frac{2c_1c_2\nu_2^2}{\nu_1^2}(\hat{\omega}^2)^2$$

is a consistent estimator of β^2. Of course, we can replace $MRQ(Y)_n$ and $MRV(Y)_n$ by $MTQ(Y)_n$ and $MBV(Y)_n$, respectively, if we want to have an estimator of β^2 which is robust to finite activity jumps.

Now we exploit the properties of stable convergence (see Podolskij (2006), Lemma 1.9) to obtain a standard central limit theorem

$$\frac{n^{\frac{1}{4}}\Big(MRV(Y)_n - \int_0^1 \sigma_u^2\, du\Big)}{\beta_n} \xrightarrow{\mathcal{D}} \mathcal{N}(0,1). \tag{3.21}$$

From the latter confidence intervals for the integrated volatility can be derived. A feasible version of Corollary 2 can be obtained similarly.

With the above formulae for β^2 and β_n^2 in hand we can choose the constants c_1 and c_2 that minimize the conditional variance. In order to compare our asymptotic variance with the corresponding results of other methods we assume that the volatility process σ is constant. In that case the conditional variance β^2 is minimized by

$$c_1 = \sqrt{\frac{18}{(c_2-1)(4-c_2)}} \cdot \frac{\omega}{\sigma} \tag{3.22}$$

$$c_2 = \frac{8}{5} \ , \tag{3.23}$$

and is equal to

$$\frac{256}{3\sqrt{18}} \cdot \sigma^3\omega \approx 20.11\sigma^3\omega.$$

Note that the limits in Corollary 1 and 2 are the same up to a constant. Consequently, the asymptotic conditional variance of $MBV(Y)_n$ is minimized for the same choice of c_1 and c_2 as above, and is approximately equal to

$$26.14\sigma^3\omega \ ,$$

when the volatility function is constant.

As already mentioned in Ait-Sahalia, Mykland & Zhang (2005) (see also Gloter & Jacod (2001a) and Gloter & Jacod (2001b)) the maximum likelihood estimator (when U is normal distributed) converges at the rate $n^{-\frac{1}{4}}$ and has an asymptotic variance

$$8\sigma^3\omega \ ,$$

which is a natural lower bound. The cubic kernel, Tukey-Hanning kernel and modified Tukey-Hanning kernel estimator which have been proposed by Barndorff-Nielsen, Hansen, Lunde & Shephard (2006) possess the asymptotic variances $9.04\sigma^3\omega$, $9.18\sigma^3\omega$ and $8.29\sigma^3\omega$, respectively. This shows that our estimator is somewhat inefficient in comparison to the listed kernel based estimators. A natural direction of future research is to modify our procedure in order to achieve a higher efficiency.

However, the concept of modulated bipower (multipower) variation has been established to provide estimates of arbitrary powers of volatility for the noisy diffusion model, which are additionally robust to finite activity jumps. These are properties which are not captured by multiscale or realised kernel approach.

For the sake of completeness we state a central limit theorem for the modulated multipower variation $MMV(Y, r_1, \ldots, r_k)_n$.

Theorem 5 *Assume that U is of the form (3.17) and condition (V) is satisfied. If M and K satisfy (2.8), and*

1. $r_1, \ldots, r_k \in (1, \infty) \cup \{0\}$ *or*

2. one of $r_i \in (0,1]$, and $\sigma_s \neq 0$ for all s,

then we have

$$n^{\frac{1}{4}}\Big(MMV(Y,r_1,\ldots,r_k)_n - MMV(Y,r_1,\ldots,r_k)\Big) \xrightarrow{\mathcal{D}_{st}} L(r_1,\ldots,r_k) \ ,$$

where $L(r_1,\ldots,r_k)$ is given by

$$L(r,l) = \sqrt{\frac{A(r_1,\ldots,r_k)}{c_1c_2}} \int_0^1 (\nu_1\sigma_u^2 + \nu_2\omega^2)^{\frac{r+l}{2}} \ dW'_u \ , \tag{3.24}$$

with

$$\begin{aligned} A(r_1,\ldots,r_k) \ &= \ \prod_{l=1}^{k} \mu_{2r_l} - (2k-1)\prod_{l=1}^{k}\mu_{r_l}^2 \\ &+ \ 2\sum_{j=1}^{k-1}\prod_{l=1}^{j}\mu_{r_l} \prod_{l=k-j+1}^{k} \mu_{r_l} \prod_{l=1}^{k-j}\mu_{r_l+r_{l+j}} \end{aligned}$$

Here W' denotes another Brownian motion defined on an extension of the filtered probability space $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\in[0,1]},P)$ and is independent of the σ-field $\mathcal{F}$.

Note that the constant $A(r_1,\ldots,r_k)$ also appears in the central limit theorem for multipower variation in a pure diffusion framework (see Barndorff-Nielsen, Graversen, Jacod, Podolskij & Shephard (2006)).

4 Simulation study

In this section, we inspect the finite sample properties of various proposed estimators for both integrated volatility and quarticity through Monte Carlo experiments. Moreover, we compare our estimators' behaviour with the properties of the corresponding kernel-based estimators from Barndorff-Nielsen, Hansen, Lunde & Shephard (2006). To this end, we choose the same stochastic volatility model as in their work, namely

$$dX_t = \mu dt + \sigma_t dW_t, \quad \sigma_t = \exp(\beta_0 + \beta_1\tau_t), \quad d\tau_t = \alpha\tau_t dt + dB_t, \quad \mathrm{corr}(dW_t, dB_t) = \rho \tag{4.1}$$

with $\mu = 0.03, \beta_0 = 0.3125, \beta_1 = 0.12, \alpha = -0.025$ and $\rho = -0.3$. U is further assumed to be normal distributed with variance ω^2.

4.1 Simulation design

We create $20,000$ repetitions of the system in equation (4.1), for which we use an Euler approximation and different values of n. Whenever we have to estimate ω^2, we choose $\hat{\omega}^2$ as defined in (3.6).

Since we state propositions for a whole class of estimators, we do not focus on one special estimator. To be precise, we investigate the finite sample properties in three different situations.

First we study the performance of $MRV(Y)_n$ as an estimator for the integrated volatility and compare it with the corresponding kernel-based statistic of Barndorff-Nielsen, Hansen, Lunde & Shephard (2006), using the modified Tukey-Hanning kernel. We denote this estimator by $KB(Y)_n$. In Table 1 we present the Monte Carlo results for both mean and variance of the two statistics for $n = 256, 1024, 4096, 9216, 16384, 25600$ and $\omega^2 = 0.01, 0.001$. Moreover, Table 2 gives the finite sample distribution of the standardised statistic in (3.21), which converges stably in law to a normal distribution. Table 3 shows the results of the asymptotic analysis of the statistic

$$\frac{n^{\frac{1}{4}}\Big(\log\Big(MRV(Y)_n\Big) - \log\Big(\int_0^1 \sigma_u^2\,du\Big)\Big)}{\beta_n / MRV(Y)_n} \xrightarrow{\mathcal{D}} \mathcal{N}(0,1), \tag{4.2}$$

which is obtained via an application of the delta method.

Secondly, we analyse the performance of the estimation of the integrated volatility in the presence of jumps. In this case we use the bipower estimator $MBV(Z)_n$, which is robust to jumps. Again, we compare its finite sample properties with the behaviour of the kernel-based estimator $KB(Z)_n$, and present the Monte Carlo results for both statistics in Table 4.

At last, we analyse how well $MRQ(Y)_n$ works as an estimator for the integrated quarticity in contrast to the proposed bipower variation estimator in Barndorff-Nielsen, Hansen, Lunde & Shephard (2006), which we call $BP(Y)_n$. Note that $BP(Y)_n$ has a convergence rate of $n^{-\frac{1}{6}}$, which is obviously slower that the convergence rate of our estimator $MRQ(Y)_n$. The Monte Carlo results for model (4.1) are given in Table 5, whereas Table 6 shows the results in the quite simple setting

$$dX_t = \mu dt + dW_t \tag{4.3}$$

with $\mu = 0.03$ as above, which we consider additionally.

As mentioned in (3.22), the asymptotic (conditional) variance of the estimators $MRV(Y)_n$ and $MBV(Y)_n$ can be minimized for an appropriate choice of c_1 and c_2, which in principal can be estimated from the data. Nevertheless, since K, M and $\frac{n}{M}$ all have to be integers, it is pretty uncertain that an optimal choice of c_1 and c_2 is feasible, when n is fixed. In practice, one should therefore estimate both IV and ω^2 from the data and choose reasonable values of c_1 and c_2, which yield feasible K and M. In these simulations the described procedure leads to $c_1 = 0.25$ for $\omega^2 = 0.01$ and $c_1 = 0.125$ for $\omega^2 = 0.001$, whereas $c_2 = 2$. Since the calculation of optimal values of c_1 and c_2 for the estimation of IQ involves the solution of polynomial equations with higher degrees than two, we have dispensed with this analysis and set $c_1 = 1$ and $c_2 = 1.6$, both for $\omega^2 = 0.01$ and $\omega^2 = 0.001$. To produce the process J we allocate one jump in the interval [0,1]. The arrival time of this jump is considered to be uniformly distributed, whereas the jump size is $N(0, h^2)$ distributed with $h = 0.1, 0.25$.

4.2 Results

Since our aim is mainly to give an idea of how well the different estimators work, we content ourselves with computing the estimated mean and variance of the bias-corrected statistics. Except for $MRV(Y)_n$ we therefore do not evaluate the accuracy of the stated central limit theorems.

Table 1 shows that $MRV(Y)_n$ works quite well as an estimator of the integrated volatility in the noisy diffusion setting, since both bias and variance are rather small, at least for sample sizes larger than $n = 1024$. For large values of n and $\omega^2 = 0.01$ it provides even better finite sample properties than $KB(Y)_n$, whereas the kernel-based estimator improves a lot, when the variance of the noise terms becomes smaller. Nevertheless, $MRV(Y)_n$ is a serious alternative to the kernel-based estimator, especially for large values of ω^2.

Table 2 indicates that the behaviour of the standardised statistic depends slightly on ω^2. For a large variance of the noise term the distribution seems to be shifted to the left, since there is a negative bias and all quantiles are overestimated. For $\omega^2 = 0.001$ the estimator's properties improve, since both bias and variance diminish. However, it has a small positive bias, whereas all quantiles are still overestimated. In both cases it takes rather large samples to provide a good approximation of a standard normal distribution. We suggest that these effects are caused by a large variance of the estimator of the integrated quarticity. A more detailed analysis of this issue is stated below.

The transition to the log-transformed statistic given by (4.2) yields an obvious improvement in the approximation of the limiting normal distribution. Table 3 shows that this statistic provides very good finite sample properties in the case of $\omega^2 = 0.01$, even for small sample sizes. For $\omega^2 = 0.001$ there is less improvement, but still the estimation of the quantiles becomes more accurate. Therefore, it is preferable to use the log-transformation in practice, when one constructs confidence sets or tests.

From Table 4 we conclude that in the noisy jump-diffusion framework the proposed bipower estimator $MBV(Z)_n$ has a much smaller bias and variance than the kernel-based statistic $KB(Y)_n$, which simply estimates the integrated volatility plus the squared jump size. Note that the negative bias for small values of n is caused by large negative bias of $MBV(Y)_n$ (which is the bipower estimator in model (2.1)) for these choices of n. We suggest that this effect somewhat compensates the impact of the jump even for moderate sample sizes.

Table 5 demonstrates the finite sample properties of $MRQ(Y)_n$ and $BP(Y)_n$ as estimates of the integrated quarticity in the noisy diffusion model. While the bias of our estimator $MRQ(Y)_n$ is much smaller than the bias of $BP(Y)_n$ for all n and ω^2, the variance of both estimators is rather large. This feature is explained by a large value of $\int_0^1 \sigma_u^8 du$ in model (4.1), which appears in the variance term for the integrated quarticity.

To reduce the impact of $\int_0^1 \sigma_u^8 du$ we present the finite sample properties of $MRQ(Y)_n$ and

$BP(Y)_n$ in Table 6 in the less complex model (4.3). We observe that the variance of $BP(Y)_n$ is smaller than that of $MRQ(Y)_n$, although $BP(Y)_n$ has a slower rate of convergence. However, we think that the efficiency of $MRQ(Y)_n$ can be improved by choosing the constants c_1 and c_2 optimally.

5 Conclusions and directions for future research

In this paper we proposed to use the modulated bipower (multipower) variation to estimate some functionals of volatility in the simultaneous presence of noise and jumps. We constructed some estimates of integrated volatility and integrated quarticity and proved their consistency. Furthermore, we showed the stable convergence of the modulated bipower variation with an optimal convergence rate $n^{-\frac{1}{4}}$. Finally, the Monte Carlo study indicates that our estimators are quite efficient at sampling frequencies normally used in applied work.

This paper highlights the potential of the modulated bipower approach, and we are convinced that many unsolved problems in a noisy (jump-)diffusion framework can be tackled by our methods. Let us mention some most important directions for future research. First, we intend to modify our approach by putting different weights on the increments of the process Y in order to obtain more efficient estimators of integrated volatility and integrated quarticity. Second, we plan to derive a multivariate version of the current approach. This can be used to estimate the quadratic covariation, which is a key concept in econometrics (see Brandt & Diebold (2006), Griffin & Oomen (2006) or Sheppard (2006)), in the presence of noise. An interesting and very important modification of this problem is the estimation of the quadratic covariation for non-synchronously observed data in the presence of noise (see Hayashi & Yoshida (2005) for more details in a pure diffusion case). Further, a joint asymptotic distribution theory for multiscale estimator (or realised kernel estimator) and the robust estimator $MBV(Y,1,1)_n$ would allow to test for finite activity jumps in a noisy jump-diffusion model.

6 Appendix

In the following we assume without loss of generality that a, σ, a', σ' and v' are bounded (for details see e.g. Barndorff-Nielsen, Graversen, Jacod, Podolskij & Shephard (2006)). Moreover, the constants that appear in the proofs are all denoted by C.

First, we show that replacing $\nu_1^{(n)}$ defined in (3.9) by ν_1 does not influence the consistency and the central limit theorem.

Lemma 1 *We have*

$$\int_0^1 (\nu_1\sigma_u^2 + \nu_2\omega^2)^{\frac{r+l}{2}}\ du - \int_0^1 (\nu_1^{(n)}\sigma_u^2 + \nu_2\omega^2)^{\frac{r+l}{2}}\ du = o_p(n^{-\frac{1}{4}})$$

for all $r, l \geq 0$.

Proof of Lemma 1 For $\frac{r+l}{2} \geq 1$ we obtain by the mean value theorem and boundedness of σ

$$\int_0^1 (\nu_1 \sigma_u^2 + \nu_2 \omega^2)^{\frac{r+l}{2}}\, du - \int_0^1 (\nu_1^{(n)} \sigma_u^2 + \nu_2 \omega^2)^{\frac{r+l}{2}}\, du \leq C(\nu_1 - \nu_1^{(n)}) = o_p(n^{-\frac{1}{4}}).$$

When $0 < \frac{r+l}{2} < 1$ we have

$$\int_0^1 (\nu_1 \sigma_u^2 + \nu_2 \omega^2)^{\frac{r+l}{2}}\, du - \int_0^1 (\nu_1^{(n)} \sigma_u^2 + \nu_2 \omega^2)^{\frac{r+l}{2}}\, du \leq (\nu_2 \omega^2)^{\frac{r+l}{2}-1}(\nu_1 - \nu_1^{(n)}) = o_p(n^{-\frac{1}{4}}) \ ,$$

and the proof is complete. □

Before we start with the proofs of main results, we introduce some more notations and prove some simple Lemmata. We consider the quantities

$$\beta_m^n = n^{\frac{1}{4}}(\sigma_{\frac{m-1}{M}} \bar{W}_m^{(K)} + \bar{U}_m^{(K)}) \qquad \beta_m^{\prime n} = n^{\frac{1}{4}}(\sigma_{\frac{m-1}{M}} \bar{W}_{m+1}^{(K)} + \bar{U}_{m+1}^{(K)}), \tag{6.1}$$

which approximate $\bar{Y}_m^{(K)}$ and $\bar{Y}_{m+1}^{(K)}$, respectively, by using the associated increments of the underlying Brownian motion W. We further define

$$\xi_m^n = n^{\frac{1}{4}} \bar{Y}_m^{(K)} - \beta_m^n \qquad \xi_m^{\prime n} = n^{\frac{1}{4}} \bar{Y}_{m+1}^{(K)} - \beta_m^{\prime n} \tag{6.2}$$

as the differences between the true quantities and their approximations. We further set $f(x) := |x|^r$ and $g(x) := |x|^l$. In the next Lemma we study the stochastic order of the terms β_m^n and ξ_m^n.

Lemma 2 *We have*

$$E[|\xi_m^n|^q] + E[|\xi_m^{\prime n}|^q] + E[|n^{\frac{1}{4}} \bar{X}_m^{(K)}|^q] < C \tag{6.3}$$

for any $q > 0$, and

$$E[|\beta_m^n|^q] + E[|\beta_m^{\prime n}|^q] + E[|n^{\frac{1}{4}} \bar{Y}_m^{(K)}|^q] < C \tag{6.4}$$

for any $0 < q < 2(r+l) + \epsilon$ with ϵ as stated in Theorem 1. Both results hold uniformly in m.

Proof of Lemma 2 We begin with the proof of (6.3). In the case $q \geq 1$ this property follows from

$$\begin{aligned} E[|\xi_m^n|^q] &= E\Big[\Big|\frac{n^{\frac{1}{4}}}{\frac{n}{M} - K + 1} \sum_{i=\frac{n(m-1)}{M}}^{\frac{nm}{M}-K} (X_{\frac{i+K}{n}} - X_{\frac{i}{n}}) - \sigma_{\frac{m-1}{M}}(W_{\frac{i+K}{n}} - W_{\frac{i}{n}})\Big|^q\Big] \qquad (6.5)\\ &\leq \frac{1}{\frac{n}{M} - K + 1} \sum_{i=\frac{n(m-1)}{M}}^{\frac{nm}{M}-K} E\Big[\Big|n^{\frac{1}{4}}((X_{\frac{i+K}{n}} - X_{\frac{i}{n}}) - \sigma_{\frac{m-1}{M}}(W_{\frac{i+K}{n}} - W_{\frac{i}{n}}))\Big|^q\Big] \\ &= \frac{1}{\frac{n}{M} - K + 1} \sum_{i=\frac{n(m-1)}{M}}^{\frac{nm}{M}-K} E\Big[\Big|n^{\frac{1}{4}}\Big(\int_{\frac{i}{n}}^{\frac{i+K}{n}} a_s ds + \int_{\frac{i}{n}}^{\frac{i+K}{n}} (\sigma_s - \sigma_{\frac{m-1}{M}}) dW_s\Big)\Big|^q\Big] \ , \end{aligned}$$

the boundedness of the functions a and σ, and a use of Burkholder's inequality. For $q < 1$ Jensen's inequality yields

$$E[|\xi_m^n|^q] \leq E[|\xi_m^n|]^q \ ,$$

and we obtain (6.3) just as above. The corresponding assertion for $n^{\frac{1}{4}}\bar{X}_m^{(K)}$ can be shown analogously.

Now let us prove (6.4). For $q \geq 1$ we have

$$E[|n^{\frac{1}{4}}\bar{Y}_m^{(K)}|^q] \leq C(E[|n^{\frac{1}{4}}\bar{U}_m^{(K)}|^q] + E[|n^{\frac{1}{4}}\bar{X}_m^{(K)}|^q])$$

Investigating the asymptotic behaviour of $\bar{U}_m^{(K)}$ it can be shown that $n^{\frac{1}{4}}\bar{U}_m^{(K)}$ can be rewritten as a weighted sum of independent random variables, for which the convergence in distribution

$$n^{\frac{1}{4}}\bar{U}_m^{(K)} \xrightarrow{\mathcal{D}} \mathcal{N}(0, \nu_2\omega^2)$$

holds. Using the continuity theorem and the moment assumption for each $0 < q < 2(r + l) + \epsilon$ we obtain by uniform integrability of $|n^{\frac{1}{4}}\bar{U}_m^{(K)}|^q$ that $E[|n^{\frac{1}{4}}\bar{U}_m^{(K)}|^q]$ is bounded. This proves (6.4) for $n^{\frac{1}{4}}\bar{Y}_m^{(K)}$. The corresponding result for the quantities β_m^n and $\beta_m^{'n}$ can be shown analogously. $\square$

The next Lemma will be used later to obtain (6.9) from (6.10). For a more general setting see Lemma 5.4 in Barndorff-Nielsen, Graversen, Jacod, Podolskij & Shephard (2006).

Lemma 3 *If*

$$Z_m^n := 1 + |\mu_m^n| + |\mu_m^{'n}| + |\mu_m^{''n}|$$

satisfies $E[|Z_m^n|^q] < C$ *for all* $0 < q < 2(r + l) + \epsilon$ *and if further*

$$\frac{1}{M}\sum_{m=1}^{M} E[|\mu_m^{'n} - \mu_m^{''n}|^2] \to 0 \tag{6.6}$$

holds, then we have

$$\frac{1}{M}\sum_{m=1}^{M} E[f^2(\mu_m^n)(g(\mu_m^{'n}) - g(\mu_m^{''n}))^2] \to 0.$$

Proof of Lemma 3 We define

$$\theta_m^n := f^2(\mu_m^n)(g(\mu_m^{'n}) - g(\mu_m^{''n}))^2$$

and

$$m_A(\delta) := \sup\{|g(x) - g(y)| : |x - y| < \delta, |x| \leq A\}.$$

For all $A > 1$ and $0 < \delta < 1$ we have

$$\begin{aligned}\theta_m^n &\leq C(A^{2r} m_A(\delta)^2 + A^{2(r+l)} 1_{\{|\mu_m'^n - \mu_m''^n| > \delta\}} + (Z_m^n)^{2(r+l)} (1_{\{|\mu_m^n| > A\}} + 1_{\{|\mu_m'^n| > A\}} + 1_{\{|\mu_m''^n| > A\}})) \\ &\leq C(A^{2r} m_A(\delta)^2 + A^{2(r+l)} \frac{|\mu_m'^n - \mu_m''^n|^2}{\delta^2} + \frac{(Z_m^n)^{2(r+l)+\epsilon'}}{A^{\epsilon'}})\end{aligned}$$

for some $\epsilon' < \epsilon$. Since $E[(Z_m^n)^{2(r+l)+\epsilon'}]$ is bounded, we obtain

$$\frac{1}{M} \sum_{m=1}^{M} E[\theta_m^n] \leq C(A^{2r} m_A(\delta)^2 + \sum_{m=1}^{M} \frac{A^{2(r+l)}}{M\delta^2} |\mu_m'^n - \mu_m''^n|^2 + \frac{1}{A^{\epsilon}}).$$

For each A we have $m_A(\delta) \to 0$. Therefore the assertion follows from (6.6). □

Proof of Theorem 1

We introduce the quantities

$$MBV^n := \sum_{m=1}^{M} \eta_m^n \qquad \text{and} \qquad MBV'^n := \sum_{m=1}^{M} \eta_m'^n,$$

where η_m^n and $\eta_m'^n$ are defined by

$$\eta_m^n := \frac{n^{\frac{r+l}{4}}}{c_1 c_2} E[|\bar{Y}_m^{(K)}|^r |\bar{Y}_{m+1}^{(K)}|^l |\mathcal{F}_{\frac{m-1}{M}}]$$

and

$$\eta_m'^n := \frac{\mu_r \mu_l}{c_1 c_2} (\nu_1 \sigma_{\frac{m-1}{M}}^2 + \nu_2 \omega^2)^{\frac{r+l}{2}},$$

respectively. Riemann integrability yields

$$\frac{1}{M} MBV'^n \xrightarrow{P} MBV(Y, r, l),$$

so we are forced to prove

$$MBV(Y, r, l)_n - \frac{1}{M} MBV^n \xrightarrow{P} 0 \tag{6.7}$$

and

$$\frac{1}{M} (MBV^n - MBV'^n) \xrightarrow{P} 0 \tag{6.8}$$

in two steps.

Considering the first step we recall the identity $\sqrt{n} = c_1 c_2 M$ and obtain therefore

$$MBV(Y, r, l)_n - \frac{1}{M} MBV^n = \sum_{m=1}^{M} \Big(\gamma_m - E[\gamma_m | \mathcal{F}_{\frac{m-1}{M}}]\Big),$$

where γ_m is given by

$$\gamma_m = n^{\frac{(r+l)}{4}-\frac{1}{2}} |\bar{Y}_m^{(K)}|^r |\bar{Y}_{m+1}^{(K)}|^l .$$

Using Lenglart's inequality (for details see Lemma 5.2 in Barndorff-Nielsen, Graversen, Jacod, Podolskij & Shephard (2006)) we find that the stochastic convergence stated in (6.7) follows from

$$\sum_{m=1}^{M} E[|\gamma_m|^2 | \mathcal{F}_{\frac{m-1}{M}}] = \frac{1}{n} \sum_{m=1}^{M} E[|n^{\frac{1}{4}} \bar{Y}_m^{(K)}|^{2r} |n^{\frac{1}{4}} \bar{Y}_{m+1}^{(K)}|^{2l} | \mathcal{F}_{\frac{m-1}{M}}] \xrightarrow{P} 0.$$

Using Hölder's inequality we find

$$\begin{aligned} & E[|n^{\frac{1}{4}} \bar{Y}_m^{(K)}|^{2r} |n^{\frac{1}{4}} \bar{Y}_{m+1}^{(K)}|^{2l} | \mathcal{F}_{\frac{m-1}{M}}] \\ \leq \; & (E[|n^{\frac{1}{4}} \bar{Y}_m^{(K)}|^{2(r+l)} | \mathcal{F}_{\frac{m-1}{M}}])^{\frac{1}{p_1}} (E[|n^{\frac{1}{4}} \bar{Y}_{m+1}^{(K)}|^{2(r+l)} | \mathcal{F}_{\frac{m-1}{M}}])^{\frac{1}{p_2}} \end{aligned}$$

with $p_1 = \frac{l}{r} + 1$ and $p_2 = \frac{r}{l} + 1$. We therefore obtain the desired result by noting that

$$E[|n^{\frac{1}{4}} \bar{Y}_m^{(K)}|^{2(r+l)}] = \mathrm{O}(1)$$

holds (uniformly in m), which is an application of Lemma 2. This completes the proof of (6.7).

To prove the assertion in (6.8) we recall that $f(x) = |x|^r$ and $g(x) = |x|^l$ and observe the identity

$$E[n^{\frac{r+l}{4}} f(\sigma_{\frac{m-1}{M}} \bar{W}_m^{(K)} + \bar{U}_m^{(K)}) g(\sigma_{\frac{m-1}{M}} \bar{W}_{m+1}^{(K)} + \bar{U}_{m+1}^{(K)}) | \mathcal{F}_{\frac{m-1}{M}}] = \frac{\mu_r \mu_l}{c_1 c_2} (\nu_1^{(n)} \sigma_{\frac{m-1}{M}}^2 + \nu_2 \omega^2)^{\frac{r+l}{2}}.$$

By Lemma 1 we obtain

$$\frac{1}{M}(MBV^n - MBV'^n) = \frac{1}{M} \sum_{m=1}^{M} E[\zeta_m^n | \mathcal{F}_{\frac{m-1}{M}}] + o_p(1)$$

with

$$\zeta_m^n = \frac{n^{\frac{r+l}{4}}}{c_1 c_2} \left(f(\bar{Y}_m^{(K)}) g(\bar{Y}_{m+1}^{(K)}) - f(\sigma_{\frac{m-1}{M}} \bar{W}_m^{(K)} + \bar{U}_m^{(K)}) g(\sigma_{\frac{m-1}{M}} \bar{W}_{m+1}^{(K)} + \bar{U}_{m+1}^{(K)}) \right).$$

To obtain the desired result it suffices to show

$$\frac{1}{M} \sum_{m=1}^{M} E[|\zeta_m^n|] \to 0.$$

We use the Cauchy-Schwarz inequality to obtain

$$\frac{1}{M} \sum_{m=1}^{M} E[|\zeta_m^n|] \leq \Big(\frac{1}{M} \sum_{m=1}^{M} E[|\zeta_m^n|^2] \Big)^{\frac{1}{2}},$$

from which we deduce that the assertion holds when

$$\frac{1}{M}\sum_{m=1}^{M} E[|\zeta_m^n|^2] \to 0. \tag{6.9}$$

In a first step we obtain for some constant $C > 0$

$$\begin{aligned} |\zeta_m^n|^2 &= \frac{1}{c_1^2 c_2^2}(f(\xi_m^n + \beta_m^n)g(\xi_{m+1}^n + \beta_{m+1}^n) - f(\beta_m^n)g(\beta_m^{\prime n}))^2 \\ &\leq C(g^2(\xi_{m+1}^n + \beta_{m+1}^n)(f(\xi_m^n + \beta_m^n) - f(\beta_m^n))^2 \\ &+ f^2(\beta_m^n)(g(\xi_{m+1}^n + \beta_{m+1}^n) - g(\beta_{m+1}^n))^2 + f^2(\beta_m^n)(g(\beta_{m+1}^n) - g(\beta_m^{\prime n}))^2)\ , \end{aligned}$$

where the quantities β_m^n and ξ_m^n are defined by (6.1) and (6.2), respectively. Since we have shown in (6.3) and (6.4) that the conditions on the boundedness of Z_m^n in our application of Lemma 3 are fulfilled, it suffices to prove

$$\frac{1}{M}\sum_{m=1}^{M} E[|\xi_m^n|^2 + |\xi_{m+1}^n|^2 + |\beta_{m+1}^n - \beta_m^{\prime n}|^2] \to 0 \tag{6.10}$$

to obtain the assertion.

For the first term in (6.10) we have

$$E[|\xi_m^n|^2] \leq \frac{1}{\frac{n}{M} - K + 1} \sum_{i=\frac{n(m-1)}{m}}^{\frac{n}{M}-K} E\Big[\Big|n^{\frac{1}{4}}((X_{\frac{i+K}{n}} - X_{\frac{i}{n}}) - \sigma_{\frac{m-1}{M}}(W_{\frac{i+K}{n}} - W_{\frac{i}{n}}))\Big|^2\Big]$$

as in (6.5). Using (2.8) and

$$(X_{\frac{i+K}{n}} - X_{\frac{i}{n}}) - \sigma_{\frac{m-1}{M}}(W_{\frac{i+K}{n}} - W_{\frac{i}{n}}) = \int_{\frac{i}{n}}^{\frac{i+K}{n}} a_s ds + \int_{\frac{i}{n}}^{\frac{i+K}{n}} (\sigma_s - \sigma_{\frac{m-1}{M}})ds$$

we obtain

$$\begin{aligned} E\Big[\Big|n^{\frac{1}{4}}((X_{\frac{i+K}{n}} - X_{\frac{i}{n}}) - \sigma_{\frac{m-1}{M}}(W_{\frac{i+K}{n}} - W_{\frac{i}{n}}))\Big|^2\Big] &\leq C\Big(n^{-\frac{1}{2}} + n^{\frac{1}{2}} E\Big[\int_{\frac{i}{n}}^{\frac{i+K}{n}} (\sigma_s - \sigma_{\frac{m-1}{M}})^2 ds\Big]\Big) \\ &\leq C\Big(n^{-\frac{1}{2}} + n^{\frac{1}{2}} E\Big[\int_{\frac{m-1}{M}}^{\frac{m}{M}} (\sigma_s - \sigma_{\frac{m-1}{M}})^2 ds\Big]\Big). \end{aligned}$$

Consequently,

$$\begin{aligned} \frac{1}{M}\sum_{m=1}^{M} E[|\xi_m^n|^2] &\leq C\sum_{m=1}^{M} E\Big[\int_{\frac{m-1}{M}}^{\frac{m}{M}} (\sigma_s - \sigma_{\frac{m-1}{M}})^2 ds\Big] + \mathrm{o}(1) \\ &= C\sum_{m=1}^{M} E\Big[\int_{\frac{m-1}{M}}^{\frac{m}{M}} (\sigma_s - \sigma_{\frac{\lfloor Ms\rfloor}{M}})^2 ds\Big] + \mathrm{o}(1) \\ &= C\int_0^1 E\Big[(\sigma_s - \sigma_{\frac{\lfloor Ms\rfloor}{M}})^2\Big] ds + \mathrm{o}(1) \end{aligned}$$

follows. Since σ is bounded and càdlàg, Lebesgue's theorem yields

$$\frac{1}{M}\sum_{m=1}^{M} E[|\xi_m^n|^2] \to 0$$

and as well for the second term in (6.10). We further have

$$\beta_{m+1}^n - \beta_m^{\prime n} = n^{\frac{1}{4}}(\sigma_{\frac{m}{M}} - \sigma_{\frac{m-1}{M}})\bar{W}_{m+1}^{(K)}.$$

Since $\bar{W}_{m+1}^{(K)}$ is independent of σ_t for any $t \le \frac{m}{M}$ we obtain

$$\begin{aligned}\frac{1}{M}\sum_{m=1}^{M} E[|\beta_{m+1}^n - \beta_m^{\prime n}|^2] &\le \frac{C}{M}\sum_{m=1}^{M} E[|\sigma_{\frac{m}{M}} - \sigma_{\frac{m-1}{M}}|^2] \\ &\le \frac{C}{M}\sum_{m=1}^{M} E[|\sigma_{\frac{m}{M}} - \sigma_s|^2 + |\sigma_s - \sigma_{\frac{m-1}{M}}|^2].\end{aligned}$$

The assertion therefore follows with the same arguments as above. That completes the proof of (6.8). □

Proof of Theorem 2

Theorem 2 can be proven by the same methods as Theorem 1. □

Proof of Theorem 4

Here we mainly use the same techniques as presented in Barndorff-Nielsen, Graversen, Jacod, Podolskij & Shephard (2006) or Christensen & Podolskij (2006b). We will state the proof of the key steps and refer to the articles quoted above for the details.

We define the quantity

$$L_n(r,l) = n^{-\frac{1}{4}}\sum_{m=1}^{M}\Big(f(\beta_m^n)g(\beta_m^{\prime n}) - E[f(\beta_m^n)g(\beta_m^{\prime n})|\mathcal{F}_{\frac{m-1}{M}}]\Big)\ , \tag{6.11}$$

where the terms β_m^n and $\beta_m^{\prime n}$ are given by (6.1), and $f(x) = |x|^r$, $g(x) = |x|^l$. In the next Lemma we state the central limit theorem for $L_n(r,l)$.

Lemma 4 *We have*

$$L_n(r,l) \xrightarrow{\mathcal{D}_{st}} L(r,l)\ ,$$

where $L(r,l)$ is defined in Theorem 4.

Proof of Lemma 4 First, note that

$$L_n(r,l) = \sum_{m=2}^{M+1}\theta_m^n + o_p(1)\ ,$$

where θ_m^n is given by

$$\begin{aligned}\theta_m^n &= n^{-\frac{1}{4}}\Big(f(\beta_{m-1}^n)\Big(g(\beta_{m-1}'^n) - \mu_l(\nu_1^{(n)}\sigma^2_{\frac{m-2}{M}} + \nu_2\omega^2)^{\frac{l}{2}}\Big) \\ &+ \mu_l(\nu_1^{(n)}\sigma^2_{\frac{m-1}{M}} + \nu_2\omega^2)^{\frac{l}{2}}\Big(f(\beta_m^n) - \mu_r(\nu_1^{(n)}\sigma^2_{\frac{m-1}{M}} + \nu_2\omega^2)^{\frac{r}{2}}\Big)\Big).\end{aligned}$$

We have that

$$E[\theta_m^n|\mathcal{F}_{\frac{m-1}{M}}] = 0\ ,$$

and

$$\sum_{m=2}^{M+1} E[|\theta_m^n|^2|\mathcal{F}_{\frac{m-1}{M}}] \xrightarrow{P} \frac{\mu_{2r}\mu_{2l} + 2\mu_r\mu_l\mu_{r+l} - 3\mu_r^2\mu_l^2}{c_1c_2}\int_0^1 (\nu_1\sigma_u^2 + \nu_2\omega^2)^{r+l}\ du.$$

Next, let $Z = W$ or B. Since θ_m^n is an even functional in W and B, and $(W,B) \overset{\mathcal{D}}{=} -(W,B)$, we obtain the identity

$$E[\theta_m^n(Z_{\frac{m}{M}} - Z_{\frac{m-1}{M}})|\mathcal{F}_{\frac{m-1}{M}}] = 0.$$

Finally, let $N = (N_t)_{t\in[0,1]}$ be a bounded martingale on $\big(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,1]}, P\big)$, which is orthogonal to W and B (i.e., with quadratic covariation $[W,N]_t = [B,N]_t = 0$ almost surely). By the arguments of Barndorff-Nielsen, Graversen, Jacod, Podolskij & Shephard (2006) we have

$$E[\theta_m^n(N_{\frac{m}{M}} - N_{\frac{m-1}{M}})|\mathcal{F}_{\frac{m-1}{M}}] = 0.$$

Now the stable convergence in Lemma 4 follows by Theorem IX 7.28 in Jacod & Shiryaev (2003). $\square$

Now we are left to prove the convergence

$$n^{\frac{1}{4}}\Big(MBV(Y,r,l)_n - MBV(Y,r,l)\Big) - L_n(r,l) \xrightarrow{P} 0. \tag{6.12}$$

Due to the result of Lemma 1 the convergence in (6.12) is equivalent to

$$\sum_{m=1}^{M} E[\vartheta_m^n|\mathcal{F}_{\frac{m-1}{M}}] \xrightarrow{P} 0\ , \tag{6.13}$$

$$\sum_{m=1}^{M} \vartheta_m'^n \xrightarrow{P} 0\ , \tag{6.14}$$

with ϑ_m^n, $\vartheta_m'^n$ defined by

$$\vartheta_m^n = n^{-\frac{1}{4}}\Big[f(n^{\frac{1}{4}}\bar{Y}_m^{(K)})g(n^{\frac{1}{4}}\bar{Y}_{m+1}^{(K)}) - f(\beta_m^n)g(\beta_m'^n)\Big]\ ,$$

$$\vartheta_m'^n = n^{\frac{1}{4}}\int_{\frac{m-1}{M}}^{\frac{m}{M}}\Big((\nu_1\sigma_u^2 + \nu_2\omega^2)^{\frac{r+l}{2}} - (\nu_1\sigma^2_{\frac{m-1}{M}} + \nu_2\omega^2)^{\frac{r+l}{2}}\Big)\ du.$$

The convergence in (6.14) has been shown in Barndorff-Nielsen, Graversen, Jacod, Podolskij & Shephard (2006), and so we concentrate on proving (6.13). Observe that

$$\vartheta_m^n = n^{-\frac{1}{4}} f(n^{\frac{1}{4}} \bar{Y}_m^{(K)}) \Big(g(n^{\frac{1}{4}} \bar{Y}_{m+1}^{(K)}) - g(\beta_m'^n) \Big) + n^{-\frac{1}{4}} g(\beta_m'^n) \Big(f(n^{\frac{1}{4}} \bar{Y}_m^{(K)}) - f(\beta_m^n) \Big)$$

Now we obtain

$$\sum_{m=1}^{M} E[\vartheta_m^n | \mathcal{F}_{\frac{m-1}{M}}] = \sum_{m=1}^{M} E[\vartheta_m^n(1) + \vartheta_m^n(2) | \mathcal{F}_{\frac{m-1}{M}}] + o_p(1) \ , \tag{6.15}$$

with $\vartheta_m^n(1)$, $\vartheta_m^n(2)$ defined by

$$\begin{aligned} \vartheta_m^n(1) &= n^{-\frac{1}{4}} \nabla g(\beta_m'^n) f(n^{\frac{1}{4}} \bar{Y}_m^{(K)}) \xi_m'^n \ , \\ \vartheta_m^n(2) &= n^{-\frac{1}{4}} \nabla f(\beta_m^n) g(\beta_m'^n) \xi_m^n \ , \end{aligned}$$

where ξ_m^n, $\xi_m'^n$ are given by (6.2), and ∇h denotes the first derivative of h. In fact, it is quite complicated to show (6.15) (especially when r or $l \in (0,1]$), but it can be proven exactly as in Barndorff-Nielsen, Graversen, Jacod, Podolskij & Shephard (2006). Note also that when r or $l \in (0,1]$ the terms $\nabla g(\beta_m'^n)$ and $\nabla f(\beta_m^n)$ are still well defined (almost surely), because $\sigma_s \neq 0$ for all s. Assumption (V) implies the decomposition

$$\xi_m^n = \xi_m^n(1) + \xi_m^n(2) \ ,$$

where $\xi_m^n(1)$, $\xi_m^n(2)$ are defined by

$$\begin{aligned} \xi_m^n(1) &= \frac{n^{\frac{1}{4}}}{\frac{n}{M} - K + 1} \sum_{i=\frac{n(m-1)}{M}}^{\frac{n}{M}-K} \Big(\int_{\frac{i}{n}}^{\frac{i+K}{n}} (a_u - a_{\frac{m-1}{M}}) du + \int_{\frac{i}{n}}^{\frac{i+K}{n}} \Big(\int_{\frac{i}{n}}^{u} a_s' ds \\ &+ \int_{\frac{i}{n}}^{u} (\sigma_{s-}' - \sigma_{\frac{m-1}{M}}') dW_s + \int_{\frac{i}{n}}^{u} (v_{s-}' - v_{\frac{m-1}{M}}') dV_s \Big) dW_u \Big) \ , \\ \xi_m^n(2) &= \frac{n^{\frac{1}{4}}}{\frac{n}{M} - K + 1} \sum_{i=\frac{n(m-1)}{M}}^{\frac{n}{M}-K} \Big(\frac{K}{n} a_{\frac{m-1}{M}} + \sigma_{\frac{m-1}{M}}' \int_{\frac{i}{n}}^{\frac{i+K}{n}} (W_u - W_{\frac{i}{n}}) dW_u \\ &+ v_{\frac{m-1}{M}}' \int_{\frac{i}{n}}^{\frac{i+K}{n}} (V_u - V_{\frac{i}{n}}) dW_u \Big) \ , \end{aligned}$$

and a similar representation holds for $\xi_m'^n$. Let us now prove that

$$\sum_{m=1}^{M} E[\vartheta_m^n(2) | \mathcal{F}_{\frac{m-1}{M}}] \xrightarrow{P} 0. \tag{6.16}$$

A straightforward application of Burkholder's inequality shows that

$$n^{-\frac{1}{4}} \sum_{m=1}^{M} E[\nabla f(\beta_m^n) g(\beta_m'^n) \xi_m^n(1) | \mathcal{F}_{\frac{m-1}{M}}] \xrightarrow{P} 0.$$

Next, note that since f is an even function ∇f is odd. Consequently, $\nabla f(\beta_m^n)g(\beta_m'^n)\xi_m^n(2)$ is an odd functional of (W,V,B). Since $(W,V,B) \stackrel{\mathcal{D}}{=} -(W,V,B)$ we obatin

$$n^{-\frac{1}{4}} \sum_{m=1}^{M} E[\nabla f(\beta_m^n)g(\beta_m'^n)\xi_m^n(2)|\mathcal{F}_{\frac{m-1}{M}}] = 0 \ ,$$

which implies (6.16). Similarly we can show that

$$\sum_{m=1}^{M} E[\vartheta_m^n(1)|\mathcal{F}_{\frac{m-1}{M}}] \xrightarrow{P} 0 \ ,$$

which completes the proof of Theorem 4. □

Proof of Theorem 5

Theorem 5 can be proven by the same methods as Theorem 4. □

References

[1] Ait-Sahalia, Y., Jacod, J. (2006). Testing for jumps in a discretely observed process. *Working paper.*

[2] Aldous, D.J., Eagleson, G.K. (1978). On mixing and stability of limit theorems. *Ann. of Prob.* **6**(2), 325-331.

[3] Andersen, T.G., Bollerslev, T., Diebold, F.X. (2006). Rouphing it up: including jump components in the measurement, modeling and forecasting of return volatility. *Forthcoming in Rev. of Econ. and Stat.*

[4] Andersen, T.G., Bollerslev, T., Diebold, F.X., Labys, P. (1998). The distribution of exchange rate volatility. *Jour. of the Amer. Stat. Ass.*, 96, 42-55.

[5] Bandi, F.M., Russel, J.R. (2004). Microstructure noise, realised variance, and optimal sampling. *Unpublished paper. Graduate School of Business, University of Chicago.*

[6] Barndorff-Nielsen, O.E., Graversen, S.E., Jacod, J., Podolskij, M., Shephard, N. (2006). *A central limit theorem for realised power and bipower variations of continuous semimartingales.* "From Stochastic Analysis to Mathematical Finance, Festschrift for Albert Shiryaev", Springer.

[7] Barndorff-Nielsen, O.E., Hansen, P.R., Lunde, A., Shephard, N. (2006). Designing realised kernels to measure the ex-post variation of equity prices in the presence of noise. *Unpublished paper.*

[8] Barndorff-Nielsen, O.E., Shephard, N. (2002). Econometric analysis of realised volatility and its use in estimating stochastic volatility models. *Jour. of Roy. Stat. Soc., Series B* 64, 253-280.

[9] Barndorff-Nielsen, O.E., Shephard, N. (2004). Power and bipower variation with stochastic volatility and jumps. *Jour. of Fin. Econ.* 2, 1-48.

[10] Barndorff-Nielsen, O.E., Shephard, N. (2006). Econometrics of testing for jumps in financial economics using bipower variation. *Jour. of Fin. Econ.* **4**(1), 1-30.

[11] Barndorff-Nielsen, O.E., Shephard, N. (2007). *Variation, jumps, market frictions and high frequency data in financial econometrics.* In R. Blundel, P. Torsten and W.K. Newey, eds, "Advances in economics and econometrics: theory and applications, ninth world congress" Cambridge University Press.

[12] Barndorff-Nielsen, O.E., Shephard, N., Winkel, M. (2006). Limit theorems for multipower variation in the presence of jumps. *Stoch. Proc. Appl.*, 116, 796-806.

[13] Black, F., Scholes, M. (1973). The pricing of options and corporate liabilities. *J. Polit. Econom.* 81, 637-654.

[14] Brandt, M.W., Diebold, F.X. (2006). A no-arbitrage approach to range-based estimation of return covariances and correlations. *Jour. of Bus.* **79**(1), 61-73.

[15] Chan, K.C., Karolyi, A.G., Longstaff, F.A., Sanders, A.B. (1992). An empirical comparison of alternative models of the short-term interest rate. *Jour. Financ.* 47, 1209-1227.

[16] Christensen, K., Podolskij, M. (2006a). Realised range-based estimation of integrated variance. *Forthcoming in Jour. of Econ.*

[17] Christensen, K., Podolskij, M. (2006b). Range-based estimation of quadratic variation. *Working paper.*

[18] Cox, J.C., Ingersoll, J.E., Ross, S.A. (1980). An analysis of variable rate loan contracts. *Jour. Financ.* 35, 389-403.

[19] Gloter, A., Jacod, J. (2001a). Diffusions with measurement errors. I - local asymptotic normality. *ESAIM: Prob. and Stat.* 5, 225-242.

[20] Gloter, A., Jacod, J. (2001b). Diffusions with measurement errors. II - measurement errors. *ESAIM: Prob. and Stat.* 5, 243-260.

[21] Griffin, J.E., Oomen, R.C.A. (2006). Covariances measurement in the presence of non-syncronous trading and market microstructure noise. *Working paper, University of Warwick.*

[22] Hansen, P.R., Lunde, A. (2006). Realised variance and market microstructure noise. *Jour. of Bus. and Econ. Stat* **24**(2), 127-161.

[23] Hayashi, T., Yoshida, N. (2005). On covariance estimation of non-synchronously observed diffusion processes. *Bernoulli* 11(2), 359-379.

[24] Huang, X., Tauchen, G. (2005). The relative contribution of jumps to total price variance. *Jour. of Fin. Econ.* **3**(4), 456-499.

[25] Jacod, J. (1994). Limit of random measures associated with the increments of a Brownian semimartingale. *Preprint number 120, Laboratoire de Probabilitiés, Univ. P. et M. Curie.*

[26] Jacod, J., Protter P. (1998). Asymptotic error distributions for the Euler method for stochastic differential equations. *Ann. of Prob.* 26, 267-307.

[27] Jacod, J., Shiryaev, A.N. (2003). *Limit theorems for stochastic processes.* Springer, Berlin.

[28] Podolskij, M. (2006). New theory on estimation of integrated volatility with applications. *PhD thesis.*

[29] Renyi, A. (1963). On stable sequences of events. *Sankya Ser.* A 25, 293-302.

[30] Sheppard, K. (2006). Realized covariance and scrambling. *Working paper, University of Oxford.*

[31] Vasicek, O. A. (1977). An equilibrium characterization of the term structure. *Jour. Financ. Econom.* 5, 177-188.

[32] Woerner, J. H. C. (2006). *Power and multipower variation: inference for high frequency data.* In "Stochastic Finance", eds. Shiryaev, A.N., do Rosarion Grossihno, M., Oliviera, P., Esquivel, M. Springer, 343-364.

[33] Zhang, L. (2004). Efficient estimation of stochastic volatility using noisy observations: a multiscale approach. *Unpublished paper: Department of Statistics, Carnegie Mellon University.*

[34] Zhang, L., Mykland, P.A., Ait-Sahalia, Y. (2005). A tale of two time scales: Determining integrated volatility with noisy high-frequency data. *Jour. Amer. Stat. Assoc.* 472, 1394-1411.

	$\omega^2 = 0.01$		$\omega^2 = 0.001$	
n	Mean	Variance	Mean	Variance
$MRV(Y)_n$				
256	0.1363	0.63	0.5245	1.782
1024	0.0433	0.219	0.1717	0.269
4096	0.0113	0.102	0.0478	0.055
9216	0.0045	0.064	0.0243	0.031
16384	0.0059	0.05	0.0129	0.021
25600	0.004	0.039	0.0094	0.017
$KB(Y)_n$				
256	-0.022	0.228	-0.0289	0.143
1024	0.0074	0.091	-0.0075	0.042
4096	0.0195	0.046	0.0004	0.015
9216	0.0203	0.038	0.001	0.009
16384	0.0201	0.04	0.001	0.007
25600	0.0178	0.046	0.0013	0.005

Table 1 *gives the Monte Carlo results for mean and variance of both* $MRV(Y)_n - \int_0^1 \sigma_u^2 du$ *and* $KB(Y)_n - \int_0^1 \sigma_u^2 du$ *for various values of* n *and* ω^2*. The data are generated from the model (4.1).*

n	Mean	Variance	0.5%	2.5%	5%	95%	97.5%	99.5%
$\omega^2 = 0.01$								
256	-0.1537	1.522	0.0438	0.0817	0.1123	0.9813	0.996	0.9999
1024	-0.107	1.208	0.0271	0.062	0.0924	0.9728	0.992	0.9995
4096	-0.1	1.124	0.02	0.0503	0.0814	0.9697	0.9887	0.9989
9216	-0.076	1.082	0.0161	0.0456	0.073	0.9653	0.9872	0.9987
16384	-0.0762	1.058	0.0139	0.0443	0.0712	0.9628	0.9861	0.9991
25600	-0.0608	1.043	0.0118	0.0398	0.068	0.9627	0.9846	0.9984
$\omega^2 = 0.001$								
256	0.0024	1.352	0.0343	0.0697	0.0994	0.9948	0.9999	1
1024	0.0875	1.114	0.0189	0.0438	0.0677	0.9744	0.9941	0.9998
4096	0.0671	1.047	0.0122	0.0361	0.059	0.9611	0.9862	0.9981
9216	0.0342	1.032	0.011	0.0323	0.0561	0.9556	0.982	0.9977
16384	0.0186	1.039	0.0103	0.0339	0.0603	0.953	0.9796	0.9972
25600	0.0066	1.049	0.0091	0.0319	0.0578	0.9527	0.9801	0.9969

Table 2 *prints mean and variance of the standardised statistic in (3.21) as well as its simulated quantiles. Precisely, the last columns give the frequency of the event that the value of the statistic lies below some typical quantiles of a standard normal distribution. The data are generated from the model (4.1).*

n	Mean	Variance	0.5%	2.5%	5%	95%	97.5%	99.5%
$\omega^2 = 0.01$								
256	0.0481	1.19	0.0073	0.0336	0.0631	0.9282	0.9614	0.9919
1024	0.0052	1.087	0.0077	0.0299	0.0582	0.9453	0.9714	0.9935
4096	-0.0103	1.046	0.0067	0.0298	0.0571	0.9484	0.9735	0.9945
9216	-0.0176	1.022	0.0066	0.0264	0.0537	0.9497	0.9745	0.994
16384	-0.019	1.025	0.0056	0.029	0.0565	0.9502	0.9757	0.9954
25600	-0.0207	1.009	0.0043	0.0257	0.0516	0.9509	0.975	0.9949
$\omega^2 = 0.001$								
256	0.2156	1.345	0.01	0.0382	0.0653	0.8957	0.9549	0.9965
1024	0.1948	1.117	0.0067	0.0267	0.0489	0.9272	0.9718	0.9969
4096	0.1266	1.072	0.0065	0.0272	0.0482	0.9364	0.9707	0.994
9216	0.0938	1.041	0.0056	0.0252	0.0482	0.9376	0.9702	0.9943
16384	0.057	1.034	0.0056	0.0253	0.0505	0.9438	0.9738	0.9946
25600	0.046	1.021	0.0057	0.0233	0.0476	0.9438	0.973	0.9948

Table 3 *prints mean and variance of the standardised statistic in (4.2) as well as its simulated quantiles. Precisely, the last columns give the frequency of the event that the value of the statistic lies below some typical quantiles of a standard normal distribution. The data are generated from the model (4.1).*

	$\omega^2 = 0.01, h = 0.25$		$\omega^2 = 0.001, h = 0.25$		$\omega^2 = 0.001, h = 0.1$	
n	Mean	Variance	Mean	Variance	Mean	Variance
256	0.0582	0.614	-0.0839	0.332	-0.1224	0.29
1024	0.0835	0.295	0.0274	0.133	-0.102	0.112
4096	0.0707	0.15	0.0466	0.063	0.0184	0.056
9216	0.0642	0.102	0.0461	0.043	0.0107	0.038
16384	0.0599	0.076	0.044	0.032	0.025	0.028
25600	0.0566	0.059	0.0415	0.025	0.0181	0.023
$KB(Y)_n$						
256	0.2227	0.39	0.2168	0.295	0.0631	0.17
1024	0.2507	0.224	0.2381	0.168	0.0924	0.063
4096	0.2667	0.173	0.249	0.142	0.102	0.038
9216	0.2675	0.17	0.2527	0.138	0.1013	0.03
16384	0.2731	0.172	0.2474	0.128	0.1009	0.028
25600	0.2674	0.177	0.2546	0.134	0.1028	0.027

Table 4 *shows mean and variance of* $MBV(Z)_n - \int_0^1 \sigma_u^2 du$ *and* $KB(Z)_n - \int_0^1 \sigma_u^2 du$*, thus in the case of jumps. We choose the sample frequency as before and analyse the finite sample properties for different values of* ω^2 *and* h*, where* h *denotes the variance of the jump size.*

	$\omega^2 = 0.01$		$\omega^2 = 0.001$	
n	Mean	Variance	Mean	Variance
$MRQ(Y)_n$				
256	0.0948	37.568	0.0976	37.282
1024	0.069	21.982	-0.058	14.315
4096	0.0271	8.587	0.0814	10.671
9216	0.0359	6.015	0.0471	5.942
16384	0.049	4.525	0.0532	4.326
25600	0.0279	3.34	0.0293	3.095
$BP(Y)_n$				
256	-1.169	8.628	-1.2835	7.595
1024	-0.6031	5.273	-0.6581	5.19
4096	-0.2556	3.286	-0.348	2.98
9216	-0.1304	2.134	-0.2024	2.031
16384	-0.0748	1.6	-0.1428	1.568
25600	0.0456	1.245	-0.1187	1.204

Table 5 *shows the finite sample properties of* $MRQ(Y)_n - \int_0^1 \sigma_u^4 du$ *and* $BP(Y)_n - \int_0^1 \sigma_u^4 du$ *in model (4.1). Both sample frequency and noise are the same as in Table 1.*

	$\omega^2 = 0.01$		$\omega^2 = 0.001$	
n	Mean	Variance	Mean	Variance
$MRQ(Y)_n$				
256	0.0745	1.348	0.0686	1.274
1024	0.0128	0.587	0.0121	0.557
4096	0.0135	0.306	0.0013	0.278
9216	0.0113	0.203	0.015	0.184
16384	0.0159	0.152	0.0155	0.14
25600	0.0088	0.117	0.0077	0.108
$BP(Y)_n$				
256	-0.2517	0.304	-0.2803	0.274
1024	-0.1811	0.186	-0.1434	0.169
4096	-0.0312	0.108	-0.0745	0.095
9216	-0.0089	0.077	-0.04	0.065
16384	0.0078	0.059	-0.0287	0.048
25600	0.0148	0.047	-0.0206	0.039

Table 6 *shows the finite sample properties of* $MRQ(Y)_n - \int_0^1 \sigma_u^4 du$ *and* $BP(Y)_n - \int_0^1 \sigma_u^4 du$ *in model (4.3). Both sample frequency and noise are the same as in Table 1.*