Optimal Portfolio Choice with Wash Sale Constraints

B. Astrup Jensen and M. Marekwica
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Abstract

In this paper we analytically solve the portfolio choice problem in the presence of wash sale constraints in a two-period model. Our results suggest that wash sale constraints can cause investors to deviate substantially from the portfolio they would otherwise have chosen. In particular we find that the optimal equity exposure changes significantly in the presence of unrealized losses. This trading behavior is to a large extent driven by the desire to realize those losses either immediately by sharply decreasing the equity exposure or indirectly by increasing the equity exposure in the first period and decreasing it again in the next period to earn the tax rebate payment. Furthermore, we show that in the presence of wash sale constraints the optimal equity exposure is not necessarily uniquely determined.

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1 Introduction

The taxable treatment of profits significantly complicates the portfolio choice problem for private investors. Realization based tax rules involve tax-timing options (see, e.g., Constantinides (1983), Constantinides (1984), Dammon, Dunn, and Spatt (1989), Dammon and Spatt (1996), Dammon, Spatt, and Zhang (2001), Klein (2001) and Dai et al. (2008)), different taxable treatment of realized capital gains versus losses (see, e.g., Ehling et al. (2009)), and tax-deferred investing (see, e.g., Shoven and Sialm (2003), Dammon, Spatt, and Zhang (2004) and Huang (2008)). In this paper we investigate a tax rule whose impact on optimal portfolio choice – to the best of our knowledge – has not received much attention so far, even though it is found in most tax codes around the world: Wash sale constraints.

In countries where wash sale rules apply, a private investor, who realizes a loss in an asset and immediately repurchases an economically equivalent asset, is treated by the tax authorities as if the loss had not been realized. These rules seek to prevent investors from trading in assets without any motive to rebalance the portfolio, but with the sole purpose of benefitting from the liquidity gain due to a tax rebate payment. Country specific rules vary with respect to the exact interpretation of “immediate” and the extent to which the repurchased asset is considered “economically equivalent” to the asset sold.

Constantinides (1983) shows that in the absence of wash sale constraints, capital losses should be realized immediately to earn tax rebate payments. Finland is usually referred to as the premier example of a country, where wash sale constraints are not binding in practice. On the other hand, in countries where wash sale constraints do apply, like the U.S., Canada and several European countries, there is a trade-off between selling the asset off to earn the tax rebate payment and leaving the equity exposure unaffected to avoid decreasing the equity exposure below an otherwise desired level.

We show that the desire to earn such a tax rebate payment can heavily affect optimal portfolio choice. Furthermore, we show that in contrast to investors trading in tax systems without wash sale constraints, the optimal portfolio for wash sale constrained investors is not necessarily uniquely determined. In particular, given a realistic parameterization of our model, we can present examples where the set of optimal solutions consists of two substantially different equity exposures.

Real world tax legislation has had recurrent problems with investors finding innovative possibilities of circumventing wash sale constraints by, e.g., trading in highly correlated assets (Stiglitz (1983)). Needless to say, legislators are persistently trying to close such loopholes in the tax legislation.\(^1\) Examples of loopholes closed include realizing losses in a taxable account and repurchasing the asset in a tax-deferred or tax-exempt environment, and transactions replacing stocks by options. To fully understand the economic consequences wash sale constraints can have for private investors’ portfolios, we focus on a setting where the investor only has access to one risky asset and one risk-free asset. This leaves no loopholes open for the investor and allows

\(^1\)See Schizer (2004) for a recent survey of such legislative initiatives.
us to isolate the economic effects caused by wash sale constraints in closed form. We follow the approach of Garlappi and Huang (2006) and focus on a two-period model with binomially distributed returns on the risky asset and an investor maximizing utility from terminal wealth. We can derive an analytical solution for optimal portfolio choice of both a wash sale constrained and an unconstrained investor with logarithmic preferences in such a model. We do so for the average basis price rule, where the investor’s historical purchase price is assumed to be the weighted average of the historical purchase prices, as well as for the exact basis price rule. For the latter, all historical purchase prices are recorded individually, implying that the marginal tax consequences from selling equity depends on the purchase price of the particular stock that the investor chooses to sell. Whereas under current US tax laws investors can use the exact basis price rule, Canadian and many European investors are subject to the average basis price rule. For the average basis price rule the derivation is less complicated. Therefore, we use this rule as a vehicle to formulate our model and solve for the optimal portfolio. Unfortunately, under the average basis price rule it is necessary to assume that the investor does not initially have unrealized capital gains or losses. For the exact basis price rule we show that such an assumption is not necessary and we derive an analytical solution irrespective of whether the investor is initially endowed with unrealized capital gains/losses or not.

Grinblatt and Keloharju (2004) note that Finnish investors, trading in a tax system in which wash sales usually do result in tax rebate payments, realize more losses than gains toward the end of the year and tend to repurchase recently sold stocks; i.e. they perform wash sales. Based on our model, we can explain why such a trading behavior is rational for investors that are not wash sale constrained, while it is irrational for investors trading in tax-systems where wash sale constraints apply. In the present financial market setting, where many private investors should be endowed with unrealized losses, wash sale constraints are likely to be binding.

In the portfolio choice literature, it is commonly assumed that wash sale constraints do not apply (see, e.g., Constantinides (1983), Constantinides (1984), Dammon, Spatt, and Zhang (2001), Dammon, Spatt, and Zhang (2004), DeMiguel and Uppal (2005), Ehling et al. (2009), Gallmeyer, Kaniel, and Tompaidis (2006), Garlappi, Naik, and Slive (2001), Huang (2008) and Hur (2001)). Our contribution is threefold. First, we show that the presence of wash sale constraints can heavily affect the investor’s portfolio choice. Second, we show that in the presence of wash sale constraints optimal portfolio policies may be discontinuous. Third, based on our analytical solution, we are even able to show that optimal equity exposure is not necessarily uniquely determined for a wash sale constrained investor. To the best of our knowledge, this paper is the first one to address these problems.

The remainder of the paper proceeds as follows. Section 2 provides an introductory example. Section 3 analyzes our model with wash sale constraints under the average basis price rule. Section 4 analyzes our model with wash sale constraints under the exact basis price rule. Section 5 summarizes the paper.
2 An introductory example

In this section we motivate the key differences between optimal portfolio choice when trading in tax systems where wash sale restrictions apply and where they do not.

For simplicity, we consider an investor with a one-period investment horizon. The investor has a portfolio consisting of 70 shares with current market price of 100 dollars per share and a historical purchase price of 120 dollars per share. I.e., the investor faces an unrealized loss of 20 dollars per share. In addition, the investor has an investment of 3000 dollars in a bank account. Hence, the investor’s total wealth is 10,000 dollars and his initial equity exposure before trading is 70%.

If the investor is not subject to wash sale constraints he would immediately realize the entire stock position in order to earn a tax rebate payment. With an assumed capital gains tax rate of 20% this would result in an immediate tax rebate payment of $70 \cdot (120 - 100) \cdot 0.2 = 280$ dollars. Realizing losses immediately is desirable as it allows the investor to invest these 280 dollars in addition to the portfolio value of 10,000 dollars. In particular, the decision to realize losses immediately does not constrain the investor’s portfolio choice.

However, in the presence of wash sale constraints the decision to realize losses and the portfolio choice problem become interrelated. The investor can only receive a tax rebate payment by reducing his equity exposure. The investor receives an immediate tax rebate payment of $(120 - 100) \cdot 0.2 = 4$ dollars per share sold. If the investor increases his equity exposure he does not receive any immediate tax rebate payment.

Within a simple binomial model with equal probabilities for pre-tax capital gains and losses of +27% and -13%, respectively, and an after-tax risk-free rate of interest of 4%, the unconstrained investor will choose to rebalance his portfolio and hold 67.5 shares worth 6750 dollars and a risk-free investment of 3530 dollars. The equity exposure is reduced from 70% to 65.66%.

The wash sale constrained investor, however, will only hold 60.3 shares worth 6030 dollars. This gives rise to an immediate tax rebate payment of only 38.8 dollars such that his risk-free investment is 4008.80. The equity exposure is reduced from 70% to 60.07%. The lower equity exposure for the wash sale constrained investor is due to the fact that he only gets a tax rebate payment by reducing his equity exposure. As earning the tax rebate payment is desirable to him, he accepts a lower equity exposure than he would otherwise have chosen. The key results of our analysis will be driven by this effect.

We now proceed to set up a model that we can solve in closed form to understand the effects of wash sale constraints in a dynamic setting.

3 The model with the average basis price rule

The basic setup of our model is inspired by Garlappi and Huang (2006). They analytically solve for the optimal asset allocation and location for a short-sale constrained log-investor having the

\[\text{Closed-form solutions for the relation between the initial equity exposure, the basis price, the current market price and the optimal equity exposure are derived in full detail in sections 3.4 and 3.5.}\]
opportunity to hold assets in both a taxable and a tax-deferred account and assume that wash sale constraints are not binding. In this paper, however, we focus on an investor having access only to a taxable account, but being subject to wash sale constraints.

3.1 Investment opportunity set

In order to isolate the impact of wash sale constraints we assume that the menu of financial assets consists of only two securities. One of them is a risk-free one-period money market account, paying an after-tax gross (net) return of $R_t$ ($r_t$) from time $t$ to time $t+1$. The other asset is a risky non-dividend paying stock. We denote the gross (net) capital gain before tax from $t$ to $t+1$ by $G_t$ ($g_t$) and assume that the return process follows a binomial distribution with probabilities of 1/2 each. The capital gains tax is paid upon realization only, and the applicable tax rate is denoted by $\tau$. By $\mu_t$ and $\sigma_t$ we denote the expected pre-tax return and standard deviation on the risky stock from time $t$ to $t+1$, such that the two possible realizations of $g_t$ are given by $g_t^+ = \mu_t + \sigma_t$ and $g_t^- = \mu_t - \sigma_t$. We assume $(1 - \tau) \frac{g_t^+ + g_t^-}{2} > r_t$, i.e. a positive after-tax risk premium, which ensures a positive demand for equity. To avoid arbitrage opportunities, we further assume that $(1 - \tau) g_t^+ > r_t > g_t^- (1 - \tau)$. Although we will specifically analyze a two-period model, the notation used is kept general and applicable for models with more than two periods. However, closed form solutions are no longer available and one would have to rely on numerical methods.

By $W_{t-}$ and $W_{t+}$ we denote the investor’s portfolio value at time $t$ before and after trading, respectively. The variable $s_t$ measures the entering value of the equity exposure at time $t$ as a fraction of $W_{t-}$. The variable $\alpha_t$ denotes the optimal exiting value of the equity exposure as a fraction of $W_{t-}$, which has to be chosen dynamically by the investor. The variable $p_t$, henceforth termed the basis price ratio, is the relation between the investor’s basis price (purchase price of the asset before trading at time $t$ used when computing realized capital gains) and the current market price at time $t$. It indicates the extent of unrealized capital gains or losses the investor is endowed with per unit of equity. A value below one refers to a situation with unrealized capital gains, and a value above one shows unrealized losses. Table 1 summarizes the definitions of our variables.

Insert Table 1 about here

3.2 The optimization problem

We consider a dynamic two-period portfolio choice model with three dates: $t = 0, 1, 2$. The investor is assumed to maximize expected utility from terminal wealth at time $t = 2$. To do so he has to dynamically choose his optimal equity exposure $\alpha_0$ and $\alpha_1$ at time $t = 0$ and $t = 1$,
respectively. Making use of the standard notation $x^+ = \max(0, x)$ his optimization problem is given by:

$$\max_{\{\alpha_0, \alpha_1\}} \mathbb{E}_0 \left[ \log \left( W_2^f \right) \right]$$

subject to

$$W_2^f = W_{2-} (1 - s_2 (1 - p_2) \tau)$$

$$W_{t+1-} = W_{t-} \left( \alpha_t (g_t - r_t) + (1 - (s_t - \alpha_t)^+ (1 - p_t) \tau) R_t \right)$$

$$p_{t+1} = \frac{s_t p_t + (\alpha_t - s_t)^+}{s_t + (\alpha_t - s_t)^+} \frac{1}{G_t}$$

$$s_{t+1} W_{t+1-} = \alpha_t G_t W_{t-}$$

According to equations (1)-(2) the investor maximizes the expected utility from terminal wealth. This involves a liquidation of all positions at the end of the investment horizon with full tax consequences. Equation (3) describes the evolution of wealth over the time interval $(t, t+1)$, including the change in wealth arising from the tax consequences of changing the equity exposure at time $t$. It indicates that there are only immediate tax consequences if the investor decreases his equity exposure.

Equation (4) shows the evolution of the basis price ratio. In case of a decrease in the equity exposure, the basis price ratio only changes as a result of the price change of the risky asset. In case there is a net increase in the equity exposure, the average purchase price after trading is a weighted average of the historical purchase price before trading and one, adjusted for the impact of the price change of the risky asset. Equation (5) describes the relation between the investor’s present optimal equity exposure ($\alpha_t$) and the entering value of the equity exposure ($s_{t+1}$) at time $t+1$.

Our optimization problem can be solved analytically by backward induction. For the logarithmic utility function it is the case that the optimal portfolio choice is independent of the wealth level at any point in time and state. Hence, we can express the investor’s indirect utility per unit of wealth as a function of his entering equity exposure $s_t$ and basis price ratio $p_t$. In a standard dynamic programming format (the “Bellman equation”) the portfolio choice problem can be expressed as

$$J_t(s_t, p_t) = \max_{\alpha_t} \mathbb{E}_t \left\{ J_{t+1} \left( s_{t+1}, p_{t+1} \right) + \log \left[ \alpha_t (g_t - r_t) + R_t \left( 1 - (s_t - \alpha_t)^+ (1 - p_t) \tau \right) \right] \right\}, \quad t = 0, 1$$

with the terminal condition

$$J_2(s_2, p_2) = \log \left[ 1 - s_2 (1 - p_2) \tau \right]$$

We turn to the impact of the exact basis price rule in section 4.
We now proceed to find the optimal portfolio choice at time $t = 1$ and time $t = 0$ by backward induction.

### 3.3 Optimal portfolio choice in the last period

Due to the different tax consequences for the investor of an increased versus a decreased equity exposure at time $t = 1$ we consider three cases. First, the investor might optimally want to increase his equity exposure ($\alpha_1 > s_1$), which defers all tax consequences to the last period. Second, the investor might optimally want to decrease the equity exposure ($\alpha_1 < s_1$), which leads to tax consequences at time $t = 1$. Third, the investor might optimally want to avoid trading in equity ($\alpha_1 = s_1$). Such an investor might want to postpone the tax consequences of selling and also be reluctant to a further increase in his equity exposure.

We now turn to the analysis of the optimal conditional equity exposure in the cases: $\alpha_1 > s_1$, $\alpha_1 < s_2$ and $\alpha_1 = s_1$ in subsections 3.3.1 to 3.3.3, respectively. In subsection 3.3.4 we show how the optimal, unconditional equity exposure can be found and that the optimal equity exposure for investors with unrealized losses is not necessarily uniquely determined.

#### 3.3.1 The case $\alpha_1 > s_1$: Increase the equity exposure

This case involves net buying. The indirect utility $J_b^1(s_1, p_1)$, conditional on $\alpha_1 > s_1$, is given by

$$J_b^1(s_1, p_1) \equiv \max_{\alpha_1} \mathbb{E}_1 \left[ \log (\alpha_1 [(1 - \tau)g_1 - r_1] + R_1 - s_1 \tau (1 - p_1)) \right]$$

The economic interpretation of this relation is that — apart from the risk premium after-tax — there is a tax consequence at time $t = 2$ of the unrealized position, but no immediate tax consequence at time $t = 1$. In addition to the tax consequences resulting from the evolution of the return of the stock throughout the second period, there is an unsettled tax benefit at time $t = 2$ if $p_1 > 1$ and an unsettled tax bill to be paid at time $t = 2$ in case $p_1 < 1$.

With equal probabilities of 1/2 each in a binomial setting, maximizing expected utility is equivalent to maximizing the product of the two possible terms. Hence, by defining

$$A^+ \equiv (1 - \tau)g_1^+ - r_1, \quad A^- \equiv (1 - \tau)g_1^- - r_1, \quad B \equiv R_1 - \tau s_1 (1 - p_1)$$

maximizing the objective function is equivalent to solving the optimization problem

$$\max_{\alpha_1} \left[ \alpha_1 A^+ + B \right] \left[ \alpha_1 A^- + B \right] = \max_{\alpha_1} \alpha_1^2 (A^+ A^-) + \alpha_1 \left[ A^+ + A^- \right] B + B^2$$

Observe that none of the expressions for $A^+$, $A^-$ and $B$ contain the choice variable $\alpha_1$. Furthermore, as already mentioned in section 3.1 it must be the case that $A^+ > 0$ and $A^- < 0$ in order to avoid arbitrage opportunities. The expression in (10) is a second order polynomial in $\alpha_1$ with
negative leading coefficient. The first order condition

$$\alpha_1 = -\frac{1}{2} \left( \frac{1}{A^+} + \frac{1}{A^-} \right) B$$

(11)

is therefore a sufficient condition for optimality. We call this solution, describing the optimal equity exposure at time $t=1$ conditional on increasing the equity exposure, $\alpha^1_b$.

While $A^+$ and $A^-$ are not affected by the equity exposure and return during the first period, it affects $B$ via $s_1$ and $p_1$. Note that for an investor subject to mark-to-market taxation – or otherwise without tax-timing opportunities – the optimal portfolio choice at time $t=1$ would be given by $\alpha_1 = -\frac{1}{2} \left( \frac{1}{A^+} + \frac{1}{A^-} \right) R_1$. The term $-\tau s_1(1-p_1)$ captures the time $t=2$ tax consequences resulting from the unrealized gains or losses at time $t=1$. Since $B$ decreases as the investor’s level of unrealized gains increases, the investor optimally holds a lower fraction of equity when being endowed with capital gains.

This reflects the fact that due to the unrealized capital gains, the investor faces a fixed tax payment at time $t=2$. This tax payment can best be hedged against by holding the risk-free bond. Consequently, an investor endowed with unrealized capital gains at time $t=1$ optimally increases his holdings in the risk-free asset and decreases his equity exposure.

Plugging (11) into (8) gives us an analytical expression for the indirect utility $J^b_1(s_1, p_1)$, conditional on $\alpha_1 > s_1$:

$$J^b_1(s_1, p_1) = \frac{1}{2} \log \left\{ B^2 \left[ \frac{1}{4} \left( \frac{1}{A^+} + \frac{1}{A^-} \right)^2 (A^+ A^-) - \frac{1}{2} \left( \frac{1}{A^+} + \frac{1}{A^-} \right) (A^+ + A^-) + 1 \right] \right\}$$

$$= \frac{1}{2} \log \left\{ B^2 \left[ 1 - \frac{1}{4} (A^+ + A^-)^2 \right] \right\} = \log(B) + \frac{1}{2} \log \left\{ 1 - \frac{1}{4} (A^+ + A^-)^2 \right\}$$

(12)

Note that only the term $\log(B)$ relates to the choice variable $\alpha_0$ at time $t=0$. The relation becomes explicit by inserting the expression in (5) for $s_1$.

### 3.3.2 The case $\alpha_1 < s_1$: Decrease the equity exposure

We now turn to an investor who decreases his equity exposure at time $t=1$. In contrast to the setting where the investor increases his equity exposure, selling equity at time $t=1$ leaves the investor with immediate tax consequences.

For such an investor, the conditional indirect utility $J^b_1(s_1, p_1)$ is given by

$$J^b_1(s_1, p_1) \equiv \max_{\alpha_1} \mathbb{E}_1 \left[ \log (\alpha_1 [(1 - \tau) g_1 - r_1 + (1 - p_1) r_1 \tau] + R_1 (1 - s_1 (1 - p_1) \tau)) \right]$$

(13)

Compared to the case where the investor increases his equity exposure, there are two key differences between equations (13) and (8). First, in the excess return on equity there is an additional term: $(1 - p_1) r_1 \tau$. It reflects the immediate marginal tax consequences from trading in equity. Second, the term $R_1 (1 - s_1 (1 - p_1) \tau)$ suggests that the investor would face all tax consequences from his unrealized capital gains or losses immediately. Taken together, these two effects imply
that an investor not holding any equity faces full immediate tax consequences. Per unit of equity he decides to hold, he can defer taxation for that unit of equity.

The solution for the optimal choice of equity exposure, cf. (11), as well as the closed-form expression for the indirect utility function, cf. (12), is similar to the former case; but the parameters $A^+, A^−$ and $B$ need to be redefined. We choose to denote these by the same letters in lower case:

\[
a^+ \equiv (1 - \tau)g_1^+ - r_1 + (1 - p_1)r_1 \tau \\
a^- \equiv (1 - \tau)g_1^- - r_1 + (1 - p_1)r_1 \tau \\
b \equiv R_1(1 - \tau s_1(1 - p_1))
\]

The optimal equity exposure at time $t=1$, conditional on selling equity, is then given by

\[
\alpha_1^* = -\frac{1}{2} \left( \frac{1}{a^+} + \frac{1}{a^-} \right) b
\]

Plugging (17) into (13) again provides us with a closed form expression for the indirect utility $J^*(s_1, p_1)$ at time $t=1$, conditional on $\alpha_1 < s_1$:

\[
J^*(s_1, p_1) = \frac{1}{2} \log \left\{ b^2 \left[ \frac{1}{4} \left( \frac{1}{a^+} + \frac{1}{a^-} \right)^2 \left( a^+ a^- \right) - \frac{1}{2} \left( \frac{1}{a^+} + \frac{1}{a^-} \right) (a^+ + a^-) + 1 \right] \right\}
= \frac{1}{2} \log \left\{ b^2 \left[ 1 - \frac{1}{4} (a^+ + a^-)^2 \right] \right\} = \log(b) + \frac{1}{2} \log \left\{ 1 - \frac{1}{4} (a^+ + a^-)^2 \right\}
\]

Again, only the term $\log(b)$ relates to the choice variable $\alpha_0$.

### 3.3.3 The case $\alpha_1 = s_1$: No change in equity exposure

Not trading in equity can be considered as limiting cases for an investor increasing or decreasing his equity exposure, respectively. This case becomes important when the two conditional solutions do not fulfill the conditions under which they were computed as we will point out in more detail throughout the following subsection. We have to treat this case separately to solve for optimal portfolio choice at time $t = 0$. This is due to the fact that the time $t = 0$ optimization problem of an investor trading in equity at time $t = 1$ differs from that of an investor who does not change his equity exposure at time $t = 1$.

The indirect utility function is obtained by plugging in $\alpha_1 = s_1$ in the second order polynomial. The two polynomials in question, the buy polynomial (10) and the analogous sell polynomial, take on the same value at the point $\alpha_1 = s_1$. It is given by

\[
J^n(s_1, p_1) = \frac{1}{2} \log \left\{ \left[ s_1 A^+ + B \right] \left[ s_1 A^- + B \right] \right\}
\]
3.3.4 The possible solutions

The optimization in the cases with a changing equity exposure concerns two second order polynomials, each with negative leading coefficient. Each of these polynomials have a unique maximum; but such a maximum is not a feasible solution if, e.g., in the buy case it calls for selling and vice versa. Additionally, both of these maxima might be irrelevant. This is the case when both the solution conditional on buying implies a decrease in the investor’s equity exposure and the solution conditional on selling suggests an increase in the investor’s equity exposure. In such a situation the investor optimally does not change his equity exposure. Depending on the two conditionally optimal equity exposures \( \alpha^b_1, \alpha^s_1 \) and the investor’s entering equity exposure \( s_1 \), we can distinguish four different cases:

\[
\alpha^*_1 = \begin{cases} 
\alpha^b_1 & \text{if } \alpha^b_1 > s_1, \, \alpha^s_1 > s_1 \quad \text{(buy)} \\
\alpha^s_1 & \text{if } \alpha^b_1 < s_1, \, \alpha^s_1 < s_1 \quad \text{(sell)} \\
s_1 & \text{if } \alpha^b_1 \leq s_1 \leq \alpha^s_1 \quad \text{(no trade)} \\
? & \text{if } \alpha^b_1 > s_1 > \alpha^s_1 
\end{cases}
\]  

(20)

In the first three cases, optimal portfolio choice can be identified immediately. The latter case is more complicated, since both conditionally optimal solutions fulfill the conditions under which they were computed. In that special case both possibilities must be checked for the value of the objective function to determine the optimal solution.

An interesting feature of this last case is that it can only appear in the presence of wash sale constraints and when \( p_1 > 1 \). A formal proof of this claim is provided in Appendix A. Intuitively, for \( p_1 \leq 1 \), in both the setting with and without wash sale constraints, the investor only has one incentive to sell equity: the motive to rebalance his portfolio in accordance with his risk preferences. Consequently, there is a unique solution when the investor is endowed with unrealized capital gains, i.e. when \( p_1 \leq 1 \). However, in case \( p_1 > 1 \), there might be two different valid conditional equity exposures. One possible investment strategy is to increase the equity exposure in order to readjust the portfolio and make up for the loss; another strategy is to decrease the equity exposure further in order to generate a tax rebate payment. Without wash sale constraints, the optimal equity exposure is uniquely determined: The investor first realizes all his unrealized capital losses, and afterwards chooses his optimal equity exposure without facing any additional tax consequences. That is, only when \( p_1 > 1 \) and in the presence of wash sale constraints, there may be both incentives to increase and decrease the equity exposure at the same time. Consequently, the problem of having two possible candidates for the optimal solution can only appear if \( p_1 > 1 \) and wash sale constraints apply.

We demonstrate in Example 1 that it may even be the case that the two possible conditional equity exposures – although pointing to very different portfolio revisions – are equally attractive in terms of the level of expected utility. I.e., the investor might be indifferent between two different equity exposures, implying that optimal portfolio choice is not necessarily uniquely determined.
Example 1
Consider the following parameters:

\[ g_1^+ = 29.5\%, \ g_1^- = -15.5\%, \ r_1 = 4\%, \ \tau = 20\%, \ s_1 = 0.5 \tag{21} \]

If the basis price ratio is \( p_1 = 1.2123 \) we find identical levels of expected utility for the buy solution \( \alpha_b^1 = 52.824\% \) and the sell solution \( \alpha_s^1 = 47.178\% \).

A more extreme example is given by the parameters:

\[ g_1^+ = 27.0\%, \ g_1^- = -13.0\%, \ r_1 = 4\%, \ \tau = 20\% \ s_1 = 0.5 \tag{22} \]

and a basis price ratio of \( p_0 = 2.2822 \). Here we find identical levels of expected utility for the buy solution \( \alpha_b^1 = 73.752\% \) and the sell solution \( \alpha_s^1 = 23.853\% \).

The third case where the investor does not trade equity can only occur when \( p_1 < 1 \). The proof of this statement is almost identical to the proof that the case with two candidates for the optimal solution can only occur when \( p_1 > 1 \), and it is also found in Appendix A. Intuitively, when the investor is endowed with unrealized capital gains, there might be two opposing effects affecting the investor’s optimal portfolio choice. On the one hand, in the absence of capital gains taxation, the investor might optimally want to decrease his equity exposure. On the other hand, as already noted by Dammon, Spatt, and Zhang (2001) among others, the investor has an incentive to defer the realization of his gains to have the opportunity of earning additional profits on the postponed tax payment. If the second incentive outweighs the first, the investor optimally does not change his equity exposure.

3.4 The optimal portfolio in the first period

Throughout this subsection we assume that the investor is not endowed with unrealized capital gains or losses at time \( t = 0 \), i.e. \( p_0 = 1 \). We generalize our results by allowing for a pre-existing unrealized capital gain or loss in section 4. At time \( t = 0 \) the investor then solves the optimization problem

\[ J_0(s_0, 1) = \max_{\alpha_0} \mathbb{E}_0 \left\{ J_1(s_1, p_1) + \log [\alpha_0 (g_0 - r_0) + R_0] \right\} \tag{23} \]

Equation (23) indicates that the investor needs to take the immediate growth of wealth as well as the state he ends up in into account. This is due to the fact that the level of unrealized capital gains or losses as well as his entering equity exposure at time \( t = 1 \) affect portfolio choice in the last period. For both the up state and the down state we therefore have to distinguish up to three cases, namely states where the investor optimally increases the equity exposure, decreases the equity exposure or leaves his equity exposure unchanged. I.e., for each of the two states and
depending upon whether the investor optimally increases, decreases or leaves his equity exposure at time \( t=1 \) unchanged, it holds that

\[
J_1(s_1, p_1) + \log [\alpha_0(g_0 - r_0) + R_0] = \begin{cases} 
\log (\alpha_0(g_0(1 - \tau) - r_0) + R_0) + \log(R_1) + (\text{constant}_1) & \text{if } \alpha_1 > s_1 \\
\log (\alpha_0(g_0(1 - \tau) - r_0) + R_0) + \log(R_1) + (\text{constant}_2) & \text{if } \alpha_1 < s_1 \\
\log [\alpha_0(g_0 - r_0) + R_0] + \frac{1}{2} \log \{[s_1 A^+ + B][s_1 A^- + B]\} & \text{if } \alpha_1 = s_1 
\end{cases}
\]  

(24)

Again, we split the solution into three steps. First, we solve the conditional portfolio choice problems at time \( t=0 \) when the investor changes his equity exposure at time \( t=1 \) in both the up state and the down state. Second, we solve the conditional cases where the investor optimally does not change his equity exposure in the up state. Third, we show how to identify optimal unconditional portfolio choice.

### 3.4.1 Optimal conditional equity exposure when trading at time \( t=1 \)

The difference between the cases where the equity exposure is changed arises solely from the timing of the tax payments. If \( \alpha_1 > s_1 \) all tax consequences are deferred to time \( t=2 \), whereas if \( \alpha_1 < s_1 \) the tax consequence of the net selling occurs immediately. If we define an effective tax rate as:

\[
\tau' = \begin{cases} 
\frac{\tau}{\frac{s_1}{R_1}} & \text{if } \alpha_1 \geq s_1 \\
\tau & \text{if } \alpha_1 < s_1 
\end{cases}
\]  

(25)

the objective function can be expressed in terms of \( \tau' \). Optimal portfolio choice at time \( t=0 \) can then be determined in a manner similar to the one used at time \( t=1 \) by maximizing a second order polynomial:

\[
\max_{\alpha_0} \left( \alpha_0 \gamma^+ + R_0 \right) \left( \alpha_0 \gamma^- + R_0 \right) = \max_{\alpha_0} \alpha_0^2 \left( \gamma^+ \gamma^- \right) + \alpha_0 R_0 \left( \gamma^+ + \gamma^- \right) + R_0^2
\]

where

\[
\gamma^\pm = \tilde{g}_0^\pm (1 - \tau') - r_0
\]  

(26)

The first order condition is similar to the first order condition at time \( t=1 \), cf. (11) and (17), and is given by

\[
\alpha_0 = -\frac{1}{2} \left( \frac{1}{\gamma^+} + \frac{1}{\gamma^-} \right) R_0
\]  

(27)

However, the solution still depends on whether the investor increases or decreases his equity exposure in the two states at time \( t=1 \) since the effective tax rate \( \tau' \) depends on whether \( \alpha_1 > s_1 \) or \( \alpha_1 < s_1 \).
3.4.2 Optimal conditional equity exposure without trading at time $t=1$

When the investor optimally does not trade at time $t=1$ with unrealized capital gains, the optimal portfolio at time $t=0$ is given by

$$
\arg \max_{\alpha_0} \left\{ \frac{1}{2} \log \left[ (\alpha_0^2 C_1 + \alpha_0 C_2 + C_3) (\alpha_0 \gamma^- + R_0)^2 \right] \right\}
$$

(28)

where

$$
F^\pm \equiv G_1^+ G_1^- (1 - \tau) + \tau
$$

$$
C_1 \equiv F^+ F^- , C_2 \equiv R_1 R_0 (F^+ + F^-) , C_3 \equiv (R_1 R_0)^2
$$

(29)

In (29), $G_1^\pm$ denotes the two possible gross returns before tax over the second period. The coefficients $F_1^\pm$ are the after-tax excess returns after two periods of a buy and hold investment strategy in the up state.

To find the optimum for the polynomial in (28) we observe that one of its extrema is $\alpha_0 = -R_0/\gamma^-$. Although it is a positive equity exposure, it is obviously an irrelevant solution, because the portfolio policy leads to bankruptcy in the down state. This is also clear from the observation that the investor’s expected utility converges to $-\infty$ as $\alpha_0$ goes to $-R_0/\gamma^-$. The first order condition provides us with three local extrema, and due to the geometry of the polynomial it is the case that

- if $C_1 > 0$ and, hence, the fourth order polynomial has a positive leading coefficient, the optimal solution is the second of the three local extrema, sorted by size
- if $C_1 < 0$ and, hence, the fourth order polynomial has a negative leading coefficient, the first and the third local extremum are both local maxima. We can rule out the largest of these, which is either $-R_0/\gamma^-$ or even larger; in both cases the down state implies bankruptcy. Hence, the optimal solution is the smallest of these two extrema.

3.4.3 Optimal unconditional portfolio choice at time $t=0$

So far we have computed optimal portfolio choices at time $t=0$ conditional on the portfolio choice at time $t=1$. However, in our dynamic setting optimal portfolio choice at time $t=1$ depends on the portfolio choice at time $t=0$. We now proceed to identify the optimal unconditional solution to the portfolio optimization problem at time $t=0$.

On each of the possible paths the slopes of the two second order polynomials, the buy polynomial as given in (10) and the analogous sell polynomial with $A^\pm$ and $B$ substituted by their lower case counterparts, are immediately calculable at the point $s_1$. The slopes of the two polynomials

---

5Recall that the no trade case can only occur in the up state.

6The intermediate steps in this derivation are found in Appendix B.
at $s_1$ are given by

\[2s_1 (A^+ A^-) + (A^+ A^-) B\]  \hspace{1cm} \text{(Buy polynomial)} \tag{31}

\[2s_1 (a^+ a^-) + (a^+ a^-) b\]  \hspace{1cm} \text{(Sell polynomial)} \tag{32}

Hence, since $s_1$ is uniquely determined by $\alpha_0$ and $g_0$, i.e. by $\alpha_0$ and the state at time $t=1$, it is immediate to check whether the assumptions behind the particular solution $\alpha_0$ considered are satisfied. E.g., if we assume a (buy in the up state, sell in the down state) solution we should have the combination of two positive slopes at $s_1$, in shorthand notation $(+,+)$, for both the buy and the sell polynomial in the up state and the combination of slopes $(-,-)$ for these two polynomials in the down state. Whenever this procedure eliminates three of the four possible combinations the solution is unique.

The situation with two possible solutions reveals itself by showing two relevant sign combinations, in which case it is necessary to compute the level of expected utility to determine which of the possible solutions is the optimal one. The case where the investor optimally does not change his equity exposure reveals itself by showing no relevant sign combinations.

The following procedure leads to the optimal portfolio choice in terms of analytic expressions:

1. Check the four cases (buy, sell) × (buy, sell). If exactly one of these four cases produces the correct sign combination, the solution is found.

2. If both (buy, buy) and (buy, sell) or (sell, buy) and (sell, sell) give correct signs we must check which policy is optimal in the down state by comparing the levels of expected utility.

3. If none of the sign tests give correct signs we are in the no trade situation in the up state. We must check which of the possibilities is optimal in the down state by examining the signs. We might still run into the situation where both signs are correct and we then need to check the level of expected utility.

### 3.5 Optimal portfolio choice in the absence of wash sale constraints

If the investor trades in a tax system that allows for wash sales, finding the optimal equity exposure is less complicated. We only have to distinguish between our three cases: buy, sell and no trade, when the investor faces capital gains at time $t=1$, i.e. $p_1<1$. If the investor has unrealized losses at time $t=1$, i.e. $p_1>1$, it is optimal to realize these losses immediately and choose the optimal equity exposure afterwards. In this case the investor’s effective tax-rate $\tau'$ is known to be equal to $\tau$, and in contrast to the setting with wash sale constraints the decision on realization of capital gains is independent from the choice of the equity exposure. Consequently, we only have to distinguish whether $\tau' = \frac{\tau}{1+\tau}$ or $\tau' = \tau$ in case the investor faces a capital gain at time $t=1$.

Without wash sale constraints the optimization problem is given by (33)-(37).
\[
\max_{\{\alpha_0, \alpha_1\}} \mathbb{E}_0 \left[ \log \left( W'_2 \right) \right] \tag{33}
\]
subject to
\[
W'_2 = W_{2-} \left( 1 - s_2 (1 - p_2) \tau \right) \tag{34}
\]
\[
W_{t+1-} = W_{t-} \left[ \alpha_t \left( g_t - r_t \right) + R_t - R_t (1 - p_t) \tau \left( 1_{\{p_t \geq 1\}} s_t + 1_{\{p_t < 1\}} \right) \right] \tag{35}
\]
\[
p_{t+1} = \begin{cases} 
\frac{1}{G_t} & \text{if } p_t \geq 1 \\
\frac{s_t p_{t+1} + (\alpha_t - s_t) + 1}{s_t + (a_t - s_t) + 1} & \text{if } p_t < 1
\end{cases} \tag{36}
\]
\[
s_{t+1}W_{t+1-} = \alpha_t G_t W_{t-} \tag{37}
\]

Equation (35) shows that in contrast to the setting with wash sale constraints (see equation (3)) all losses are realized immediately. We can again simplify this optimization problem by expressing the investor’s optimal portfolio choice in terms of per unit entering wealth.

As mentioned above, in the case without wash sale constraints it suffices to distinguish between whether the investor has a basis price ratio below or above one. If \( p_1 < 1 \), the investment problem is the same as for a wash-sale constrained investor. For \( p_1 \geq 1 \), however, the investor gets an immediate full tax rebate payment. His optimal equity exposure \( \alpha_1 \) at time \( t=1 \) is then found as the solution to the optimization problem

\[
\arg \max_{\alpha_1} \mathbb{E}_1 \left[ \log \left[ \alpha_1 \left( g_1 (1 - \tau) - r_1 \right) + R_1 (1 - s_1 \tau (1 - p_1)) \right] \right] \tag{38}
\]

With the notation already introduced this solution is identical to

\[
\arg \max_{\alpha_1} \left[ \alpha_1 A^+ + b \right] \cdot \left[ \alpha_1 A^- + b \right] = \arg \max_{\alpha_1} \left[ \alpha_1^2 A^+ A^- + \alpha \left( A^+ + A^- \right) b + b^2 \right] \tag{39}
\]

In explicit form the optimal equity exposure \( \alpha_{1w} \) for the case \( p_1 \geq 1 \) is then given by

\[
\alpha_{1w} = -\frac{1}{2} \left( \frac{1}{A^+} + \frac{1}{A^-} \right) b \tag{40}
\]

This combination of parameters does not occur in the case with wash sale constraints; it is a mixture of the parameters for the buy and the sell solutions, indicating that the investor receives the full tax rebate payment immediately and independent of the portfolio choice.

Analogous to (12) and (18) the indirect utility function at time \( t=1 \) can be expressed as
\[ J_1(s_1, p_1) = \begin{cases} 
\log(b) + \frac{1}{2} \log \left( 1 - \frac{1}{4} \left( \frac{A^+ + A^-}{A^+ A^-} \right)^2 \right) & \text{if } p_1 \geq 1 \\
\log(B) + \frac{1}{2} \log \left( 1 - \frac{1}{4} \left( \frac{A^+ + A^-}{A^+ A^-} \right)^2 \right) & \text{if } p_1 < 1, \quad \alpha_i^b > s_1 \\
\log(b) + \frac{1}{2} \log \left( 1 - \frac{1}{4} \left( \frac{a^+ + a^-}{a^+ a^-} \right)^2 \right) & \text{if } p_1 < 1, \quad \alpha_i^s < s_1 \\
\frac{1}{2} \log \left( (s_1 A^+ + B)(s_1 A^- + B) \right) & \text{if } p_1 < 1, \quad \alpha_i^s \geq s_1 \geq \alpha_i^b 
\end{cases} \]

Except for the case where \( p_1 \geq 1 \) this is identical to the case of a wash sale constrained investor. The case \( p_1 \geq 1 \) only occurs in a situation where \( g_0 = g_0^- \). If we impose the property that \( p_1 \geq 1 \) iff \( g_0 = g_0^- \) the optimization problem simplifies a little\(^7\) but we still do not know in advance in which state we end when \( g_0 = g_0^+ \) and \( p_1 < 1 \).

We proceed to find the optimal portfolio at time \( t = 0 \) in the same way as for the wash sale constrained case. The only difference arises from the exact expression for the indirect utility in the down state, but the solution procedure is exactly the same.

### 3.6 Comparison of the scenarios

Having derived analytical expressions for optimal portfolio choice, both with and without wash sale constraints, we next turn to an illustration of how wash sale constraints affect optimal asset allocation. In this section, we illustrate the impact of wash sale constraints on the investor’s optimal asset allocation in the last period by comparing optimal portfolio choice for investors facing wash sale constraints and investors that are not wash sale constrained. Throughout our numerical examples we assume a tax rate on capital gains of \( \tau = 20\% \), a pre-tax expected return on the risky asset of \( \mu = 7\% \) and a standard deviation of \( \sigma = 20\% \). The risk-free rate of interest after tax is set to \( r = 4\% \). Given that the qualitative effects are robust to a variation of these parameters, we only report results for this particular choice of parameters.

**Insert Figure 1 about here**

The upper graph in Figure 1 depicts optimal portfolio choice for an investor facing wash sale constraints, the lower graph depicts optimal asset allocation for an investor who is not wash sale constrained. It indicates that both the investor’s basis price ratio as well as his entering equity exposure heavily affect portfolio choice for investors that are wash sale constrained; while the entering equity exposure only has a strong impact for unconstrained investors in case these have large unrealized capital gains.

Investors confronted with a basis price ratio below one, i.e. investors endowed with unrealized capital gains at the beginning of the last period, are confronted with the same relation between their optimal equity exposure on the one hand and their basis price ratios as well as their entering

\(^7\) In case of a positive risk-free rate of interest it may still be the case that \( p_1 < 1 \) even when \( g_0 = g_0^- \). However, if the parameters are such that this is the case there is little insight to be gained from the absence of wash sale constraints.
equity exposures on the other hand, in tax systems with and without wash sale constraints. This is due to the fact that given unrealized capital gains, the wash sale constraint is not binding to the investor’s portfolio choice problem and therefore does not affect his optimal equity exposure in the last period.

For an investor with unrealized capital gains the optimal policy for small values of $s_1$ is to buy and increase the equity exposure. The small negative slope reflects that as $s_1$ increases, so does the tax bill to be paid at the final date, which mimicks a short position in a non-interest bearing bond. Consequently, the investor adjusts the portfolio in accordance with this by increasing the position in the risk-free bond. The graph is composed of straight lines as revealed in the expression for the optimal equity exposure (cf. (9) and (11)):

$$\alpha^t_1 = -\frac{1}{2} \left( \frac{1}{A^r} + \frac{1}{A} \right) (R_1 - \tau s_1 (1 - p_1))$$

When $p_1 < 1$ the slope is negative. As $s_1$ increases, both graphs show an upward sloping region reflecting a change in the optimal policy to a “no trade in equity” policy. The exiting equity exposure in both settings with and without wash sale constraints rises above the level that the investor would have chosen in a situation, where there are no existing tax consequences to take into account. Such a situation arises either because $p_1 = 1$ or, for comparison, because the tax code has eliminated the timing options as, e.g., in a mark-to-market based taxation system.

However, at some point the entering equity exposure $s_1$ becomes too large in order for the “no trade in equity” policy to be in accordance with the basic risk preferences. This is where the graphs show the next kink and the optimal policy changes to a “sell equity” in order to reduce the equity exposure. As a consequence of selling equity the investor is now forced to pay capital gains taxes. To mitigate the capital gains tax payments, the investor only slightly reduces his equity exposure — a finding which has also been noted in the numerical studies of Dammon, Spatt, and Zhang (2001), Gallmeyer, Kaniel, and Tsmpaidis (2006) and Ehling et al. (2009), among others.

If, however, the investor is endowed with unrealized capital losses at the beginning of the second period, wash sale constraints are binding and therefore affect the investor’s optimal portfolio choice. An investor who does not face wash sale constraints optimally realizes his losses immediately to receive the tax rebate payment; in this way the implicit long position in a non-interest bearing bond is at least converted into an interest bearing position. Even better, the proceeds are reinvested optimally. For a wash sale constrained investor this no longer holds since wash sale constraints induce a trade-off between realizing losses to earn the tax rebate payment immediately and not decreasing the equity exposure too sharply. While in the absence of wash sale constraints the investor can choose his equity exposure after trading independent from his realization of losses, a wash sale constrained investor can only realize a loss by decreasing the equity exposure.

Figure 1 indicates that while a wash sale constrained investor’s entering equity exposure is small, the investor optimally avoids realizing his losses and increases his equity exposure for not ending
up with a portfolio containing too little equity. If, however, the investor’s initial equity exposure is big, so is the amount of the investor’s unrealized capital gains. In such a situation the investor optimally sells at least some part of his equity to earn the tax rebate payments. The amount of equity the investor sells depends crucially on the amount of unrealized capital losses. The higher the investor’s unrealized capital losses per unit of equity, that is, the higher the investor’s basis price ratio, the higher the tax rebate payments per unit of equity and consequently, the lower the investor’s optimal equity exposure after trading.

The jump in the surface of the wash sale constrained investor’s optimal equity exposure is due to the fact that given substantial amounts of unrealized capital losses, the investor ends up in the case where both \( \alpha^b_1 \) and \( \alpha^s_1 \) are possible solutions for the optimal equity exposure. At the level of the jump, the expected utility from an equity exposure of \( \alpha^b_1 \) is identical to the expected utility from the equity exposure of \( \alpha^s_1 \), cf. Example 1. Since \( \alpha^b_1 \) and \( \alpha^s_1 \) can differ substantially – especially if the investor’s basis price ratio is very big – this results in a jump in the surface describing the optimal equity exposure for a wash sale constrained investor.\(^8\)

3.7 Pre-existing basis price ratio

We have so far assumed that \( p_0 = 1 \). A more general version of the model arises, if we assume that there is a pre-existing basis price ratio \( p_0 \neq 1 \), implying that the investor may already be endowed with unrealized capital gains or losses to take into consideration when finding the optimal equity exposure \( \alpha_0 \).

The main complication with this generalization is that with the average basis price rule, \( p_1 \) is a weighted average of \( p_0 \) and 1, implying that \( p_1 \) becomes a function of \( \alpha_0 \). With \( p_0 = 1 \) this weighting is irrelevant. The same is true with either \( s_0 = 0 \) or \( s_0 = 1 \), because one of the weights will be zero. But solving for the optimal portfolio at time \( t = 0 \) in this generalized setting requires finding roots of polynomials of order up to 11.\(^9\) Regrettably, there are no closed-form solutions available for that type of problem.

To generalize our results, we next modify our model and make use of the exact share identification rule for computing the investor’s realized gains and losses instead of the average basis price rule. This avoids the averaging of basis prices and thereby enables us to solve the two period model in closed form without having to make simplifying assumptions about the pre-existing basis price ratio in the first period.

4 The model with the exact basis price rule

Using the exact share identification rule we no longer have to assume that \( p_0 = 1 \). We instead assume that the investor is endowed with some initial equity exposure \( s_0 \in [0,1] \) and an arbitrary initial basis price ratio \( p_0 \).

\(^8\)In a general equilibrium setting, such discontinuities might be a source of potential instability in the financial markets.

\(^9\)The formal derivations are available from the authors upon request.
If we increase our initial equity exposure \( s_0 \) at time \( t=0 \) there are two entering equity exposures at time \( t=1 \) with two different tax bases. In the following, \( s_{1,0} \) denotes that part of the entering equity exposure at time \( t=1 \) that originates from the pre-existing equity exposure \( s_0 \) at time \( t=0 \). The basis price ratio for this position is \( p_{1,0} = p_0/G_0 \). \( s_{1,1} \) denotes that part of the entering equity exposure at time \( t=1 \) that originates from net buying at time \( t=0 \). Its basis price ratio is \( p_{1,1} = 1/G_0 \).

Similar notation is used at time \( t=2 \), where there can be up to three possible sources of entering equity exposure. \( s_{2,0} \) denotes that part of the entering equity exposure at time \( t=2 \) that originates from the pre-existing equity exposure at time \( t=0 \). The associated basis price ratio is \( p_{2,0} = p_0/(G_0G_1) \). \( s_{2,1} \) denotes that part of the entering equity exposure at time \( t=2 \) that originates from net buying at time \( t=0 \). The corresponding basis price ratio is \( p_{2,1} = 1/(G_0G_1) \). Finally, \( s_{2,2} \) denotes that part of the entering equity exposure at time \( t=2 \) that originates from net buying at time \( t=1 \). The basis price ratio for this is \( p_{2,2} = 1/G_1 \).

We summarize this additional notation in Table 2.

Insert Table 2 about here

4.1 The optimal portfolio in the last period

Analogous to the optimization problem in (1)-(5) we search for the optimal equity exposure in the last period:

\[
\max_{\alpha_1} \mathbb{E}_1 \left[ \log \left( W'_2 \right) \right] \quad (43)
\]

subject to

\[
W'_2 = W_{2-} \left( 1 - \sum_{i=0}^{2} s_{2,i} (1 - p_{2,i}) \tau \right) \quad (44)
\]

\[
W_{2-} = \alpha_1 (g_1 - r_1) + (1 - TP_1) R_1 \quad (45)
\]

\( TP_1 \) is the tax payment caused by a change in equity exposure at time \( t=1 \). In case the investor sells equity, it is always optimal to sell the equity with the highest basis price ratio first as this results in the highest tax rebate payments or the lowest tax payments. If we assume that \( p_{1,1} > p_{1,0} \), equivalent to \( p_0 < 1 \), it is optimal to sell from the equity bought at time \( t=0 \) first; and if it is optimal to sell more than \( s_{1,1} \), we next sell from the equity pre-existing at time \( t=0 \). And analogously for the case where \( p_{1,1} \leq p_{1,0} \), equivalent to \( p_0 \geq 1 \), we first sell from the equity pre-existing at time \( t=0 \). Hence, we can express the tax payment \( TP_1 \) as

\[
TP_1 = \tau \left[ (s_{1,0} + s_{1,1} - \alpha_1)^+(1 - p_{1,max}) + (s_{1,0} - \alpha_1)^+(p_{1,1} - p_{1,0})^+ + (s_{1,1} - \alpha_1)^+(p_{1,0} - p_{1,1})^+ \right] \quad (46)
\]

where \( p_{1,max} \equiv \max\{p_{1,0}; p_{1,1}\} \).
In comparison to the situation under the average basis price rule the number of optimal conditional portfolio choice problems to compute increases, because the case of an investor selling equity branches into (i) the situation where only a part of the position with the highest basis price ratio is sold, (ii) the situation where also part of the equity position with the lowest basis price ratio is sold and (iii) the limiting situation where the entire equity position with the highest basis price ratio is sold, but none of the equity position with the lowest basis price ratio is sold.

Instead of (at most) three conditional equity exposures in the model with the average basis rule, we now have five conditional equity exposures to compute.

The details of the solution procedure is very similar to the procedure developed previously for the average basis price rule. We can reduce most of the optimization problem to a problem of optimizing one or more second order polynomials with negative leading coefficient. In the most demanding cases fourth order polynomials are involved, whose roots can be expressed in closed form using the Cardano formula. The previously defined $A/a$ and $B/b$ parameters carry over with minor modifications. In all cases we obtain analytical solutions for the possible candidates and can check their feasibility by using the slope test developed in section 3.4.3. If there is more than one locally optimal solution we again have to pick the correct one by comparing the levels of expected utility which are also available in closed form for these equity exposures.

The formal derivations are similar to the solution procedure already described in detail for the average basis price rule. For this reason we have placed the details in Appendix C. As for the main text we proceed to discuss the impact of wash sale constraints on optimal portfolio choice under the exact basis price rule, both at time $t = 1$ and at time $t = 0$.

### 4.2 The optimal equity exposure at time $t = 1$

In the presence of wash sale constraints there is a trade-off between receiving a tax rebate from a position carrying an unrealized loss and choosing the correct equity exposure in accordance with the investor’s risk attitude.

Under the exact share identification rule the optimal portfolio choice at time $t = 1$ depends on the entering equity exposures $s_{1,0}$ and $s_{1,1}$ and their respective basis price ratios $p_{1,0}$ and $p_{1,1}$.

In Figure 2 we consider an investor whose entering equity exposure bought at time $t = 0$, i.e. $s_{1,1} = 20\%$. The two graphs show the optimal equity exposure at time $t = 1$ as a function of the pre-existing equity position $s_{1,0}$ and its basis price ratio $p_{1,0}$, the latter varying between 0 and 2. Since $s_{1,1} = 20\%$ we vary $s_{1,0}$ between 0 and 80\% due to the assumed short-selling constraint. In the upper graph we show the case with an unrealized loss for the position $s_{1,1}$, and in the lower graph we show the case with an unrealized gain for the position $s_{1,1}$.

With $p_{1,0}$ carrying an unrealized loss, i.e. $p_{1,0} > 1$, we observe that the optimal equity exposures in the two graphs behave very similar. The functional relation is also qualitatively very similar.
to the situation analyzed previously for the average basis price rule in Figure 1. Initially, with \( s_{1,0} \) close to zero, the investor is way out of line with his risk attitude and wants to increase his equity exposure to a level of 60-65%. At the same time the investor has the unrealized loss from the \( s_{1,0} \). The unrealized loss is a claim on the tax authority mimicking a long position in a non-interest bearing bond. In the presence of wash sale constraints, the only way to turn this claim into an interest bearing asset is to reduce the equity exposure.

Hence, when \( s_{1,0} \) increases the latent tax rebate from the unrealized position is equivalent to an increase in such a bond position, which is balanced by an increase in the equity exposure. However, this effect is small with the parameters chosen as the implicit bond holding is only a small fraction of the investor’s wealth. At some level of \( s_{1,0} \) the opportunity cost of not earning interest on this implicit bond holding becomes large enough to make it optimal to immediately reduce an otherwise desired equity exposure. This is the discontinuity observed around the level 40% for \( s_{1,0} \). Naturally, the discontinuity point is decreasing with an increasing level of \( p_{1,0} \). The drop in equity exposure is of the same order of magnitude as the drop observed in Figure 1 for the average basis price rule.

With \( p_{1,0} \) carrying an unrealized gain, i.e. if \( p_{1,0} < 1 \), the graphs differ depending on whether \( s_{1,1} \) carries an unrealized loss or an unrealized gain.

In the upper graph the investor has an incentive to realize (some of) the loss stemming from \( s_{1,1} \). When \( s_{1,0} \) increases from 0 on, the investor again wants to increase his equity exposure to a level of 60-65%. At some level of \( s_{1,0} \) the entering equity exposure is sufficiently high to make it optimal to immediately reduce the equity exposure in order to earn the tax rebate on some of the \( s_{1,1} \) position. At some level of \( s_{1,0} \) the entering equity exposure is so high that the entire \( s_{1,1} \) position is optimally realized. After this point the investor optimally wants to avoid tax payments and therefore allows for higher equity exposures when \( s_{1,0} \) is large.

In the lower graph both positions carry an unrealized gain when \( p_{1,0} < 1 \). At approximately the same level of \( s_{1,0} \), i.e. at around 40%, it now becomes optimal for the investor to cease buying more equity. When \( s_{1,0} \) increases beyond this level the investor could reduce his \( s_{1,1} \) position.

However, in contrast to the upper graph the investor is no longer receiving a tax rebate for doing so, but rather has to pay capital gains taxes. With unrealized capital gains, it is therefore optimal to accept a certain deviation from the otherwise desired equity exposure. The magnitude of deviation the investor is prepared to accept depends on the level of the immediate marginal tax consequences of reducing the equity exposure.

In our setting the investor is endowed with two different tax bases. The investor optimally sells part of the position that has the highest basis price ratio first. In contrast to the average basis price rule, the lower graph shows a plateau for \( p_{1,0} < p_{1,1} \) in this region, where the investor is selling from the \( s_{1,1} \) position to counteract the higher level of \( s_{1,0} \). At approximately \( s_{1,0} = 65% \) the entire \( s_{1,1} \) position has been sold off and the marginal tax consequences of selling equity jumps upwards. This causes the investor to accept further deviations from the otherwise desired equity exposure.
For $p_{1,0}$ between $p_{1,1}$ and 1 the first up hill part of the graph disappears when $s_{1,0}$ increases. The investor optimally wants to retain the $s_{1,1}$ position and only adjust the equity exposure via selling some of the $s_{1,0}$ position.

### 4.3 No wash sale constraints

In the absence of wash sale constraints the optimization problem for the last period only changes as far as the tax payments $TP_1$ are concerned. The tax consequences now depend crucially on whether $p_{1,j}$ is above or below one; in the former case it is always optimal to realize the position that carries a loss and then optimize the equity exposure afterwards. If we again look at the case $p_{1,1} \geq p_{1,0}$, equivalent to $p_0 < 1$, we can express the tax consequences by

$$TP_1 = \left\{ \begin{array}{ll}
1_{\{p_{1,1} > 1\}} 
\left[ s_{1,1} (1 - p_{1,1}) + (1_{\{p_{1,0} > 1\}} s_{1,0} + 1_{\{p_{1,0} \leq 1\}} (s_{1,0} - \alpha_1)^+) (1 - p_{1,0}) \right]

+ 1_{\{p_{1,1} \leq 1\}} \tau 
\left[ (s_{1,0} + s_{1,1} - \alpha_1)^+ (1 - p_{1,1}) + (s_{1,0} - \alpha_1)^+ (p_{1,1} - p_{1,0})^+ + (s_{1,1} - \alpha_1)^+ (p_{1,0} - p_{1,1})^+ \right]
\end{array} \right.$$  

(47)

However, if $p_0 > 1$ it is optimal to realize all unrealized losses at time $t=0$ and start from scratch with the wealth level increased by the immediate tax rebate. I.e., the basis price ratio is brought back to unity and at time $t=1$ there is only one basis price ratio, $p_{1,1}$, and $s_{1,0} = s_{2,0} = 0$. The indirect utility function at time $t=1$ is then identical to the one already derived under the average basis price rule, cf. (41).

### 4.4 The optimal portfolio at time $t=0$

Having discussed the impact of wash sale constraints on optimal portfolio choice in the last period, we next turn to its impact on portfolio choice in the first period. Compared to the last period, the dynamic nature of the portfolio problem gives rise to additional effects.

Insert Figure 3 about here

Figure 3 shows the impact of wash sale constraints in the first period. The upper part shows the relation between the optimal equity exposure on the one hand and the initial equity exposure as well as the investor’s initial basis price ratio on the other hand for an investor who is facing wash sale constraints. The lower panel depicts our results for an investor who is not facing wash sale constraints.

In line with our previous for the last period it shows that portfolios are most affected when investors are endowed with unrealized losses, i.e. $p_0 > 1$.

For an investor who is not facing wash sale constraints our results are rather similar to those of Figure 1; i.e., the length of the remaining investment horizon does not have a strong impact on conditional portfolio choice for such an investor. However, in the presence of wash sale constraints the length of the remaining investment horizon substantially affects optimal portfolio choice for
investors with unrealized losses. This difference results from the fact that with a remaining investment horizon of more than one period the investor has to take both the immediate tax consequences and possible future tax consequences of his portfolio choice into account. Hence, it can make sense to deviate from the equity exposure an investor would have chosen in the absence of unrealized losses. An immediate decrease in his equity exposure leaves the investor with an immediate tax rebate payment. Even though an immediate increase in his present equity exposure does not leave the investor with an immediate tax rebate payment, it allows the investor to sharply reduce his equity exposure in the next period to benefit from tax rebate payments in that period. This strategy, ceteris paribus, becomes more desirable as the investor’s present level of unrealized losses per unit of equity increases.

In Figure 4 we show a cross-section of Figure 3 where the pre-existing equity exposure $s_0$ is fixed at 20%. For an investor with an extraordinary level of unrealized gains per unit of stock held ($p_0 \approx 0$) the optimal equity exposure is approximately 62%. When the level of unrealized capital gains decreases, i.e. moving along the section labeled I, the investor slightly increases his equity exposure. The reason for this is that the tax liability, mimicking a short position in a non-interest bearing bond, shrinks. The optimal portfolio choice at time $t=1$ is not to trade in the up state and to further increase the equity position in the down state. As $p_0$ increases beyond unity, i.e. the tax liability becomes a tax claim, the investor soon reaches a point where the optimal equity exposure becomes discontinuous in $p_0$ and jumps to the section labeled II. The reason for this jump is that the investor’s optimal trading strategy at time $t=1$ changes; in both states the investor now chooses to realize some part of the pre-existing equity position. To avoid ending up with an undesirably low equity exposure at time $t=1$ the investor chooses a higher equity exposure at time $t=0$ he can sell from at time $t=1$. The jump does not occur immediately when $p_0$ increases beyond the value of 1. The reason for this is that in order for the investor to be able to harvest the tax rebate payment in the down state, he has to accept either a tax payment or an undesirably high equity exposure in the up state. For small values of unrealized losses this disadvantage outweighs the desirability of the tax rebate payment. As the level of the investor’s unrealized losses further increases, so does the desirability of realizing larger parts of them at time $t=1$. The investor does not want to end up with a too low equity exposure after trading at time $t=1$ and therefore his optimal time $t=0$ equity exposure increases with the increasing level of the unrealized losses.

Further increasing the level of unrealized losses, the investor reaches the point where the entire pre-existing equity position $s_{10}$ is optimally sold in the downstate. Simultaneously, we reach the level of $p_0$ that turns the tax payment in the up state into a tax rebate payment. This is marked by the kink between regions II and III. As the level of $p_0$ increases beyond this point the investor wants to harvest more and more tax losses in the up state, which is why he further increases
his equity exposure at time $t=0$. The decrease in the slope is due to the fact that he then has exhausted the opportunities for receiving further tax rebate payments in the down state.

The last part of the curve marked IV reflects outcomes in which he has also exhausted the opportunities for tax rebate payments in the up state. The optimal trading strategy at time $t=1$ is now to sell off the entire pre-existing equity position $s_{1,0}$ in both states. The small remaining positive slope is due to the fact that the implicit bond holding keeps growing with $p_0$ in this region.

5 Summary and conclusion

In this paper we analytically solve for the optimal portfolio choice for both a wash sale constrained and an unconstrained investor. We examined the effect of wash sale constraints for investors trading under both the average and the exact basis price rule. The latter allowed us to generalize our results and added a variety of different tax effects.

Our results show that equity exposures significantly affected by wash sale constraints in the presence of unrealized losses, because there are two opposing desires affecting portfolio choice. First, the investor might want to increase the equity exposure to rebalance the portfolio in accordance with his risk attitude. Second, the investor might want to decrease his equity exposure to earn tax rebate payments. The existence of two counteracting motives causes the portfolio choice problem to be a non-concave optimization problem with the possibility of local optima.

We showed that for a wash sale constrained investor it is possible to have two global optima, implying that the optimal equity exposure is not necessarily uniquely determined. This might also complicate the use of numerical methods, which in itself makes an analytically solvable model attractive.

The existence of wash sale constraints may cause the investor to choose an equity exposure that substantially deviates from the equity exposure that would follow solely from risk attitude considerations in the absence of tax effects. The investor has an incentive to accelerate tax rebate payments by either realizing losses immediately and accepting a lower equity exposure or by increasing his present equity exposure in order to be able to realize more losses in the forthcoming period. We demonstrated that the optimal portfolio choice is not everywhere a continuous function of the level of unrealized losses; i.e., a small change in the amount of unrealized loss can result in a large change in the optimal equity exposure.

Our results suggest promising issues for further research. Except for Grinblatt and Keloharju (2004), there is so far only limited empirical evidence on the effect of wash sale constraints on investor behavior. Additionally, although a far-reaching topic, it would be interesting to explore the impact of wash sale constraints on asset pricing in a general equilibrium setting.
Appendices

A Proofs

Proof that \( \alpha_s^1 < s_1 < \alpha_b^1 \) can only occur for \( p_1 > 1 \). The case \( \alpha_s^1 < s_1 < \alpha_b^1 \) can only occur if the slope of the buy polynomial is positive at \( s_1 \) and the slope of the sell polynomial is negative at \( s_1 \). This is equivalent to the conditions:

\[
2s_1(A^+A^-) + (A^+ + A^-)B > 0 \\
2s_1(a^+a^-) + (a^+ + a^-)b < 0
\]

(A.1)  

(A.2)

Inserting the relations

\[
a^\pm = A^\pm + r_1(1 - p_1) \equiv A^\pm + V \\
b = B - r_1(1 - p_1)s_1 \equiv B - Vs_1
\]

(A.3)  

(A.4)

into (A.2) we get the equivalent expressions

\[
2s_1(A^+A^-) + (A^+ + A^-)B > 0 \\
-2s_1(A^+A^- + (A^+ + A^-)V + V^2) - (A^+ + A^- + 2V)(B - Vs_1) > 0
\]

(A.5)  

(A.6)

Adding these two relations gives the following necessary condition:

\[
(-V)(s_1(A^+ + A^-) + 2B) > 0
\]

(A.7)

As long as the risk premium after-tax is positive, i.e. \( \frac{1}{2}(A^+ + A^-) > 0 \), this holds if and only if \( V < 0 \). I.e. if and only if \( p_1 > 1 \).

Proof that \( \alpha_s^1 > s_1 > \alpha_b^1 \) can only occur for \( p_1 < 1 \). We now prove that the case where the investor does not want to trade in equity can only occur when there are unrealized capital gains. The proof is identical in form to the proof given above.

The case \( \alpha_s^1 < s_1 < \alpha_b^1 \) can only occur if the slope of the buy polynomial is negative at \( s_1 \) and the slope of the sell polynomial is positive at \( s_1 \). This is equivalent to the conditions:

\[
2s_1(A^+A^-) + (A^+ + A^-)B < 0 \\
2s_1(a^+a^-) + (a^+ + a^-)b > 0
\]

(A.8)  

(A.9)

Inserting the relations (A.3)-(A.4) into (A.9) we get the equivalent expressions

\[
2s_1(A^+A^-) + (A^+ + A^-)B < 0 \\
-2s_1(A^+A^- + (A^+ + A^-)V + V^2) - (A^+ + A^- + 2V)(B - Vs_1) < 0
\]

(A.10)  

(A.11)
Adding these two relations gives the following necessary condition:

\[-V(s_1(A^+ + A^-) + 2B) < 0\]  \hspace{1cm} (A.12)

As long as the risk premium after-tax is positive, i.e. \(\frac{1}{2}(A^+ + A^-) > 0\), this holds if and only if \(V > 0\). I.e. if and only if \(p_1 < 1\).

\section*{B Relation (30)}

The intermediate steps for the result in (30) are given here. Recall that by definition \(\alpha_0 G_0 = s_1(\alpha_0(g_0 - r_0) + R_0)\).

\[
\begin{align*}
J_1(\alpha_0, g_0) + \log (\alpha_0(g_0 - r_0) + R_0) = \\
\frac{1}{2} \left[ \log \{ (s_1 A^+ + B)(\alpha_0(g_0 - r_0) + R_0) \} + \log \{ (s_1 A^- + B)(\alpha_0(g_0 - r_0) + R_0) \} \right] = \\
\frac{1}{2} \left[ \log \{ (s_1 A^+ + R_1 - \tau s_1(1 - p_1))(\alpha_0(g_0 - r_0) + R_0) \} + \right. \\
\left. \log \{ (s_1 A^- + R_1 - \tau s_1(1 - p_1))(\alpha_0(g_0 - r_0) + R_0) \} \right] = \\
\frac{1}{2} \left[ \log \{ (\alpha_0 G_0 A^+ + R_1 R_0 + R_1 \alpha_0(g_0 - r_0) - \tau \alpha_0 g_0) \} + \right. \\
\left. \log \{ (\alpha_0 G_0 A^- + R_1 R_0 + R_1 \alpha_0(g_0 - r_0) - \tau \alpha_0 g_0) \} \right] = \\
\frac{1}{2} \left[ \log \{ (\alpha_0 F^+ + R_1 R_0)(\alpha_0 F^- + R_1 R_0) \} \right] \hspace{1cm} (B.1)
\end{align*}
\]
C The exact basis price rule

C.1 The optimization problem at time $t=1$ with wash sale constraints

We can reformulate the optimization problem in (43)-(46) for the last period as\textsuperscript{10}

$$\max \mathbb{E}_1 \left[ \log(H_1) + \log \left( 1 - \tau \sum_{i=0}^{2} s_{2,i} (1 - p_{2,i}) \right) \right]$$

$$1 - p_{2,i} = \frac{G_{1} - p_{1,i}}{G_1} \quad i = 0, 1$$

$$1 - p_{2,2} = \frac{g_1}{G_1}$$

$$s_{2,0} = \frac{G_1}{H_1} \begin{cases} 
    s_{1,0} & \text{if } s_{1,0} < \alpha_1 \land p_{1,1} > p_{1,0} \\
    \alpha_1 & \text{if } \alpha_1 \leq s_{1,0} \land p_{1,1} > p_{1,0} \\
    \alpha_1 - s_{1,1} & \text{if } s_{1,1} < \alpha_1 \land p_{1,1} \leq p_{1,0} \\
    0 & \text{if } \alpha_1 \leq s_{1,1} \land p_{1,1} \leq p_{1,0}
\end{cases}$$

$$s_{2,1} = \frac{G_1}{H_1} \begin{cases} 
    s_{1,1} & \text{if } \alpha_1 > s_{1,0} + s_{1,1} \\
    \alpha_1 - s_{1,0} & \text{if } s_{1,0} < \alpha_1 \leq s_{1,0} + s_{1,1} \land p_{1,1} > p_{1,0} \\
    0 & \text{if } \alpha_1 \leq s_{1,0} \land p_{1,1} > p_{1,0} \\
    s_{1,1} & \text{if } s_{1,1} < \alpha_1 \leq s_{1,0} + s_{1,1} \land p_{1,1} \leq p_{1,0} \\
    \alpha_1 & \text{if } \alpha_1 \leq s_{1,1} \land p_{1,1} \leq p_{1,0}
\end{cases}$$

$$s_{2,2} = \frac{G_1}{H_1} \left( \alpha_1 - s_{1,0} - s_{1,1} \right)^+$$

$$H_1 = \alpha_1 (g_1 - r_1) + R_1 (1 - TP_1)$$

$$TP_1 = \tau \left[ (s_{1,0} + s_{1,1} - \alpha_1)^+ (1 - p_{1,\text{max}})^+ + (s_{1,0} - \alpha_1)^+ (p_{1,1} - p_{1,0})^+ + (s_{1,1} - \alpha_1)^+ (p_{1,0} - p_{1,1})^+ \right]$$

For each of the possible scenarios regarding the basis price ratio, i.e. $p_{11} > p_{10}$ and $p_{11} \leq p_{10}$, respectively, this looks like three different cases. For $p_{11} > p_{10}$, e.g., these cases are $\alpha_1 \leq s_{1,0}$, $s_{1,0} < \alpha_1 \leq s_{1,0} + s_{1,1}$ and $\alpha_1 > s_{1,0} + s_{1,1}$, respectively. However, the borderline cases $\alpha_1 = s_{1,0}$ and $\alpha_1 = s_{1,0} + s_{1,1}$ require special attention. The latter situation corresponds to the “do nothing” case from the average basis price rule. The former one involves selling off exactly the additional equity bought at time $t=0$. Reaching a boundary in the domain of possible solutions normally involves a discontinuous change in the slope of the objective function, and these points must therefore be examined separately. Again, if the optimal solution at time $t=0$ does not involve net buying the problem simplifies.

\textsuperscript{10}In case one of the equity positions is not defined, the corresponding basis price ratio is not defined either. To simplify the notation we just interpret the sum by omitting undefined terms.
C.2 The different cases

Case 1: Buy at time $t = 1$

This case is very similar to the buy case with the average basis price rule, cf. equations (9)–(12). There are no tax consequences at time $t = 1$, i.e. $TP_1 = 0$; and the previously defined parameters $A^\pm$ carry over to this situation in unchanged form. The $B$ parameter needs a minor adjustment to accomplish the fact that there will be two equity exposures – $s_{1,0}$ and $s_{1,1}$ – that need to be tracked individually in case the investor has increased the equity exposure at time $t = 0$ beyond what the pre-existing exposure inherited from the past:

$$B \equiv R_1 - (s_{1,0}(1 - p_{1,0}) + s_{1,1}(1 - p_{1,1})) \tau$$  \hspace{1cm} (C.2)

The solution for the indirect utility function is now

$$\max_{\alpha_1} \mathbb{E}_1 \left[ \log \left\{ \alpha_1 (g_1(1 - \tau) - r_1) + R_1 - (s_{1,0}(1 - p_{1,0}) + s_{1,1}(1 - p_{1,1})) \tau \right\} \right] = \max_{\alpha_1} \mathbb{E} \left[ \log \left\{ (\alpha_1 g_1(1 - \tau) - r_1 + B) \right\} \right] = \max_{\alpha_1} \frac{1}{2} \log \left\{ (\alpha_1 A^+ + B)(\alpha_1 A^- + B) \right\}$$  \hspace{1cm} (C.3)

Inserting the optimal value of $\alpha_1$ we again have a closed form solution for the indirect utility function:

$$J_{1b}^p(s_{1,0}, p_{1,0}; s_{1,1}, p_{1,1}) = \max_{\alpha_1} \frac{1}{2} \log \left\{ (\alpha_1 A^+ + B)(\alpha_1 A^- + B) \right\}$$

$$= \log(B) + \frac{1}{2} \log \left\{ 1 - \frac{1}{4} \frac{(A^+ + A^-)^2}{A^+ A^-} \right\}$$  \hspace{1cm} (C.4)

Case 2: Sell from the most favorable position only at time $t = 1$

This case is also similar to the sell case with the average basis price rule, cf. equations (13)–(18). However, it is slightly more involved. It is still the case, though, that the $a^\pm$ parameters are independent of the choice variable, but we need to distinguish the “sell little” case from the “sell much” case below, where not only the equity with the most favorable tax status is sold, but also part of the equity with the less favorable tax status is sold. We therefore denote the $a^\pm$ in this case by $a_{1l}^\pm$. Furthermore, the $b$ parameter must be redefined as $b_1$:

$$a_{1l}^\pm = g_1(1 - \tau) - r_1 + r_1(1 - p_{1,\text{max}}) \tau$$  \hspace{1cm} (C.5)

$$b_1 = R_1 (1 - (s_{1,0} + s_{1,1})(1 - p_{1,\text{max}}) \tau) - \tau \left( s_{1,0}(p_{1,1} - p_{1,0})^+ + s_{1,1}(p_{1,0} - p_{1,1})^+ \right)$$  \hspace{1cm} (C.6)

where $p_{1,\text{max}} \equiv \max\{p_{1,0}; p_{1,1}\}$. Given this the indirect utility function is again found by maxi-
mizing a 2nd order polynomial with negative leading coefficient:

\[
\max_{\alpha_1} \mathbb{E} \left[ \log \left\{ \alpha_1 (g_1(1 - \tau) - r_1 + r_1(1 - p_{1,\text{max}})\tau) + R_1 \left( 1 - (s_{1,0} + s_{1,1})(1 - p_{1,\text{max}})\tau \right) - \tau (s_{1,0}(p_{1,1} - p_{1,0})^+ + s_{1,1}(p_{1,0} - p_{1,1})^+) \right\} \right] = \\
\max_{\alpha_1} \mathbb{E} \left[ \log \left\{ \alpha_1 (g_1(1 - \tau) - r_1 + r_1(1 - p_{1,\text{max}})\tau) + b_l \right\} \right] = \max_{\alpha_1} \frac{1}{2} \log \left[ (\alpha_1 a_1^+ + b_l)(\alpha_1 a_1^- + b_l) \right]
\]  
(C.7)

Inserting the optimal value of \( \alpha_1 \) we again have a closed form solution for the indirect utility function:

\[
J_1^I(s_{1,0}, p_{1,0}; s_{1,1}, p_{1,1}) = \max_{\alpha_1} \frac{1}{2} \log \left\{ (\alpha_1 a_1^+ + b_l)(\alpha_1 a_1^- + b_l) \right\} \\
= \log(b_l) + \frac{1}{2} \log \left\{ 1 - \frac{1}{4} \left( \frac{a_1^+ + a_1^-}{a_1^+ a_1^-} \right)^2 \right\}
\]  
(C.8)

**Case 3: Sell from both equity positions at time \( t=1 \)**

Again, parameters need to be redefined just as above, but the procedure is entirely analogous:

\[
a_m^\pm = g_1(1 - \tau) - r_1 + r_1(1 - p_{1,\text{min}})\tau \\
b_m = R_1 \left( 1 - (s_{1,0}(1 - p_{1,0}) + s_{1,1}(1 - p_{1,1}))\tau \right)
\]  
(C.9, C.10)

The optimization problem for this case is and the resulting closed form solution for the indirect utility function become:

\[
\max_{\alpha_1} \mathbb{E} \left[ \log \left\{ \alpha_1 (g_1(1 - \tau) - r_1 + r_1(1 - p_{1,\text{min}})\tau) + R_1 \left( 1 - (s_{1,0}(1 - p_{1,0}) + s_{1,1}(1 - p_{1,1}))\tau \right) \right\} \right] = \\
\max_{\alpha_1} \mathbb{E} \left[ \log \left\{ \alpha_1 (g_1(1 - \tau) - r_1 + r_1(1 - p_{1,\text{min}})\tau) + b_m \right\} \right] = \\
\max_{\alpha_1} \frac{1}{2} \log \left[ (\alpha_1 a_m^+ + b_m)(\alpha_1 a_m^- + b_m) \right]
\]  
(C.11)

Inserting the optimal value of \( \alpha_1 \) we the closed form solution for the indirect utility function becomes:

\[
J_1^m(s_{1,0}, p_{1,0}; s_{1,1}, p_{1,1}) = \max_{\alpha_1} \frac{1}{2} \log \left\{ (\alpha_1 a_m^+ + b_m)(\alpha_1 a_m^- + b_m) \right\} \\
= \log(b_m) + \frac{1}{2} \log \left\{ 1 - \frac{1}{4} \left( \frac{a_m^+ + a_m^-}{a_m^+ a_m^-} \right)^2 \right\}
\]  
(C.12)

**Case 4: No change in equity at time \( t=1 \)**

As with the average basis price rule this occurs when the buy polynomial and the “sell little” polynomial, intersect at \( \alpha_1 = s_{1,0} + s_{1,1} \) with “wrong slopes”.

In this case the investor keeps all equity, i.e. \( \alpha_1 = s_1 \equiv s_{1,0} + s_{1,1} \), and the indirect utility function
becomes:

\[ J_i^n(s_{1,0}, p_{1,0}; s_{1,1}, p_{1,1}) = \frac{1}{2} \log \left\{ (s_{1} A^+ + B)(s_{1} A^- + B) \right\} \]
\[ = \frac{1}{2} \log \left\{ s_{1}^2 A^+ A^- + s_{1} B(A^+ + A^-) + B^2 \right\} \]  \hspace{1cm} (C.13)

Case 5: Sell the most favorable position in full at time \( t = 1 \)

With the exact basis price rule there is an additional situation, where two second order polynomials meet, namely the combination of the “sell little” polynomial with the “sell much” polynomial. Here the investor sells exactly the full amount \( s_{1,1} \) if \( p_{1,1} > p_{1,0} \); and, vice versa, exactly the full amount \( s_{1,0} \) if \( p_{1,1} < p_{1,0} \). For the former situation we have

\[ J_i^m(s_{1,0}, p_{1,0}; s_{1,1}, p_{1,1}) = \frac{1}{2} \log \left\{ (s_{1,0} a_i^+ + b_i)(s_{1,0} a_i^- + b_i) \right\} \]
\[ = \frac{1}{2} \log \left\{ s_{1,0}^2 a_i^+ a_i^- + s_{1,0} b_i(a_i^+ + a_i^-) + b_i^2 \right\} \]  \hspace{1cm} (C.14)

And analogously for the latter situation.

C.3 The decision problem at time \( t = 0 \)

Having found the indirect utility function at time \( t = 1 \) for all cases, we now proceed to find the optimal portfolio choice \( \alpha_0 \) at time \( t = 0 \). The utility optimization problem to solve is contingent on whether the investor buys or sells equity at time \( t = 0 \). The important distinction here is whether there is one or two basis price ratio(s) to consider at time \( t = 1 \), so the sell case includes the situation where the entering and the exciting equity exposure at time \( t = 0 \) are identical.

If it is optimal to buy at time \( t = 0 \), i.e. \( \alpha_0 > s_0 \), there are no immediate tax consequences, so the dynamic programming formulation of the optimization problem is given as

\[ J_0(s_0, p_0) = \max_{\alpha_0} \mathbb{E} \left[ J_1(s_{1,0}, p_{1,0}; s_{1,1}, p_{1,1}) \right] + \log \left\{ \alpha_0(g_0 - r_0) + R_0 \right\} \]  \hspace{1cm} (C.15)

And the links between the entering exposures \( s_{1,0} \) and \( s_{1,1} \) and \( \alpha_0 \) become

\[ s_{1,0} = \frac{s_0 G_0}{\alpha_0 (g_0 - r_0) + R_0}, \quad s_{1,1} = \frac{(\alpha_0 - s_0) G_0}{\alpha_0 (g_0 - r_0) + R_0} \]  \hspace{1cm} (C.16)

\[ s_{1,0}(1 - p_{1,0}) = \frac{s_0 (G_0 - p_0)}{\alpha_0 (g_0 - r_0) + R_0}, \quad s_{1,1}(1 - p_{1,1}) = \frac{(\alpha_0 - s_0) g_0}{\alpha_0 (g_0 - r_0) + R_0} \]  \hspace{1cm} (C.17)

If the investor sells at time \( t = 0 \), i.e. \( \alpha_0 \leq s_0 \), the situation is equivalent in structure to the average basis price rule, for which we already know the indirect utility function. In this case there is only one basis price at time \( t = 1 \), namely the pre-existing one, and \( s_{1,1} = 0 \). However, the expressions for \( A^\pm, a^\pm, B \) and \( b \) are different. The link between the entering exposure \( s_{1,0} \)
and $\alpha_0$ becomes

$$s_{1,0} = \frac{\alpha_0 G_0}{\alpha_0 (g_0 - r_0) + R_0 (1 - (s_0 - \alpha_0)(1 - p_0)\tau)} \quad (C.18)$$

$$s_{1,0}(1 - p_{1,0}) = \frac{\alpha_0 (G_0 - p_0)}{\alpha_0 (g_0 - r_0) + R_0 (1 - (s_0 - \alpha_0)(1 - p_0)\tau)} \quad (C.19)$$

In order to formulate the decision problem at time $t=0$ it is necessary to split the analysis into the two cases:

- a net buy decision at time $t=0$
- a net sell decision at time $t=0$, including no change in the equity exposure as a borderline case

For the net buy situation the 25 possible combinations of optimal decisions at time $t=1$ must be compared. Among these many will lead to infeasible solutions in the sense that they do not fulfill the slope test. However, given the non-convexity of the overall optimization problem there is likely to be local optima that need to be compared directly via a comparison of the level of expected utility. In this case the number of such possible local optima can be up to three in the worst case.

For the net sell situation there are only 9 possible combinations of optimal decisions at time $t=1$ to evaluate, and the number of local optima can each be at most two.
References


Figure 1: This figure depicts the relation between the investor’s optimal equity exposure and his basis price ratio $p_1$ as well as his entering equity exposure $s_1$ in the last period under the average basis price rule. The upper graph shows this relation for a wash sale constrained investor, the lower graph for an investor who is not facing wash sale constraints. In both graphs we assume the parameters to be given by $\tau = 20\%$, $r_1 = 4\%$, $\mu_1 = 7\%$ and $\sigma_1 = 20\%$. 
Exact basis price rule at time $t = 1$

With wash sale constraints, $t = 1$, $s_{1,1} = 0.2$, $p_{1,1} = 1.1494$

With wash sale constraints, $t = 1$, $s_{1,1} = 0.2$, $p_{1,1} = 0.7874$

Figure 2: This figure depicts the relation in the last period between the investor's optimal equity exposure and the combination of his basis price ratio $p_{1,0}$ and the level of pre-existing equity exposure $s_{1,0}$ under the exact basis price rule. That part of the investor's initial equity coming with basis price ratio $p_{1,1}$ is $s_{1,1} = 20\%$. The upper graph shows this relation for the case of $p_{1,1} = 1.1494$, the lower graph for $p_{1,1} = 0.7874$, corresponding to the basis price ratios from a one period loss or gain, respectively. In both graphs we assume the parameters to be given by $\tau = 20\%$, $r_1 = 4\%$, $\mu_1 = 7\%$ and $\sigma_1 = 20\%$. 
Exact basis price rule at time $t=0$

With wash sale constraints, $t=0$

Without wash sale constraints, $t=0$

Figure 3: This figure depicts the relation between the investor’s optimal equity exposure and his basis price ratio $p_0$ as well as his initial equity exposure $s_0$ in the first period under the exact basis price rule. The upper graphs show this relation for a wash sale constrained investor, the lower graph for an investor who is not facing wash sale constraints. In both graphs we assume the parameters to be given by $\tau = 20\%$, $r = 4\%$, $\mu = 7\%$ and $\sigma = 20\%$. 
Figure 4: This figure depicts the relation between the investor’s optimal equity exposure and his basis price ratio $p_0$ at time $t=0$ for an investor endowed with an initial equity exposure $s_0 = 20\%$. We assume the parameters driving portfolio choice to be given by $\tau = 20\%$, $r = 4\%$, $\mu = 7\%$ and $\sigma = 20\%$. 
<table>
<thead>
<tr>
<th>Variable</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$W_{t-}$</td>
<td>Investor’s wealth level before trading at time $t$ (entering wealth at time $t$)</td>
</tr>
<tr>
<td>$W_{t+}$</td>
<td>Investor’s wealth level after trading at time $t$ (exiting wealth at time $t$)</td>
</tr>
<tr>
<td>$s_t$</td>
<td>Entering equity exposure at time $t$ before trading expressed as a fraction of total wealth $W_{t-}$ before trading</td>
</tr>
<tr>
<td>$\alpha_t$</td>
<td>Exiting equity exposure at time $t$ after trading expressed as a fraction of total wealth $W_{t-}$ before trading</td>
</tr>
<tr>
<td>$g_t$</td>
<td>Net return on equity from time $t$ to $t+1$</td>
</tr>
<tr>
<td>$G_t$</td>
<td>Gross return on equity ($G_t = 1 + g_t$)</td>
</tr>
<tr>
<td>$r_t$</td>
<td>After tax net return on the risk-free asset from time $t$ to $t+1$</td>
</tr>
<tr>
<td>$R_t$</td>
<td>After tax gross return on the risk-free asset ($R_t = 1 + r_t$)</td>
</tr>
<tr>
<td>$p_t$</td>
<td>The investor’s basis price ratio before trading at time $t$</td>
</tr>
<tr>
<td>$\tau$</td>
<td>The tax rate on capital gains realized at time $t$</td>
</tr>
</tbody>
</table>

Table 1: Definition of variables
Entering equity exposure at time $t$, $t = 1, 2$, before trading expressed as a fraction of total wealth $W_{t-}$ before trading. The second subscript “0” denotes equity pre-existing at time $t = 0$.

Entering equity exposure at time $t$, $t = 1, 2$, before trading expressed as a fraction of total wealth $W_{t-}$ before trading. The second subscript “1” denotes equity bought at time $t = 0$, but not pre-existing at time $t = 0$.

Entering equity exposure at time $t = 2$, before trading expressed as a fraction of total wealth $W_{2-}$ before trading. The second subscript “2” denotes equity bought at time $t = 1$, but not pre-existing at neither time $t = 0$ nor time $t = 1$.

The investor’s basis price ratio before trading at time $t$ for that part of equity bought at time $j$:

<table>
<thead>
<tr>
<th>$s_{t,0}$</th>
<th>$s_{t,1}$</th>
<th>$s_{2,2}$</th>
<th>$p_{t,j}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Entering equity exposure at time $t$, $t = 1, 2$, before trading expressed as a fraction of total wealth $W_{t-}$ before trading. The second subscript “0” denotes equity pre-existing at time $t = 0$.</td>
<td>Entering equity exposure at time $t$, $t = 1, 2$, before trading expressed as a fraction of total wealth $W_{t-}$ before trading. The second subscript “1” denotes equity bought at time $t = 0$, but not pre-existing at time $t = 0$.</td>
<td>Entering equity exposure at time $t = 2$, before trading expressed as a fraction of total wealth $W_{2-}$ before trading. The second subscript “2” denotes equity bought at time $t = 1$, but not pre-existing at neither time $t = 0$ nor time $t = 1$.</td>
<td>The investor’s basis price ratio before trading at time $t$ for that part of equity bought at time $j$: $p_{1,0} = \frac{p_0}{\sigma_0}$, $p_{1,1} = \frac{1}{\sigma_0}$, $p_{2,0} = \frac{p_0}{\sigma_0}\sigma_1$, $p_{2,1} = \frac{1}{\sigma_0}\sigma_1$, $p_{2,2} = \frac{1}{\sigma_1}$</td>
</tr>
</tbody>
</table>

Table 2: Definition of variables
2. B.J. Christensen and M. Ørregaard Nielsen (March 2006), The implied-realized volatility relation with jumps in underlying asset prices.
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